

Entropy on abelian groups

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Abstract

We extend to endomorphism of arbitrary abelian groups the definition of the algebraic entropy h given by Peters for automorphisms and we study the properties of h . In particular, we prove the Addition Theorem for h and we obtain a Uniqueness Theorem for h in the category of all abelian groups and their endomorphisms. The third of our main results is the Bridge Theorem connecting the algebraic entropy and the topological entropy by the Pontryagin duality.

1 Introduction

Inspired by the notion of entropy invented by Clausius in thermodynamics in the fifties of the nineteenth century, Shannon introduced the notion of entropy in Information Theory by the end of the forties of the last century. A couple of years later, Kolmogorov and Sinai introduced the notion of (measure) entropy in ergodic theory. By appropriate modification of their definition, Adler, Konheim, and McAndrew [1] obtained the notion of topological entropy $h_{top}(\psi)$ of a continuous self-map $\psi : X \rightarrow X$ of a compact topological space X (see Section 7 for the definition).

The compact groups and their continuous endomorphisms provide an instance where both the measure and the topological entropy can be applied. Indeed, every compact group admits a translation invariant (Haar) measure. Moreover, as noticed by Halmos [9] a continuous endomorphism $\psi : K \rightarrow K$ of a compact group K is measure preserving if and only if ψ is surjective. It was established by Berg that for a surjective continuous endomorphism $\psi : K \rightarrow K$ of a compact group K the measure entropy and the topological entropy coincide. As far as non-surjective endomorphisms $\psi : K \rightarrow K$ are concerned, the measure entropy of ψ is not defined (as ψ is not measure preserving), while $h_{top}(\psi)$ still makes sense. However, as observed in [26], the restriction of ψ to the subgroup $\text{Im}_\infty \psi = \bigcap_{n \in \mathbb{N}} \psi^n(K)$ is surjective and $h_{top}(\psi) = h_{top}(\psi|_{\text{Im}_\infty \psi})$ (see Lemma 7.6). This allows us to say that in this sense the topological entropy and the measure entropy of the continuous endomorphisms of compact groups coincide.

Using ideas briefly sketched in [1], Weiss [30] developed the definition of algebraic entropy for endomorphisms of abelian groups as follows. Let G be an abelian group and let $\phi : G \rightarrow G$ be an endomorphism of G . For a finite subgroup F of G and $n \in \mathbb{N}$, let $T_n(\phi, F) = F + \phi(F) + \dots + \phi^{n-1}(F)$ be the n -th ϕ -trajectory of F (while $T(\phi, F) = \sum_{n \in \mathbb{N}} \phi^n(F)$ is the ϕ -trajectory of F). Moreover, the limit (which is the *algebraic entropy of ϕ with respect to F*)

$$H(\phi, F) = \lim_{n \rightarrow \infty} \frac{\log |T_n(\phi, F)|}{n} \quad (1.1)$$

exists. The *algebraic entropy* of ϕ is

$$\text{ent}(\phi) = \sup\{H(\phi, F) : F \leq G \text{ finite}\}. \quad (1.2)$$

According to the main result of Weiss [30], the topological entropy of a continuous endomorphism $\psi : K \rightarrow K$ of a profinite abelian group coincides with the algebraic entropy of the adjoint map $\hat{\psi} : \hat{K} \rightarrow \hat{K}$ of ψ , where \hat{K} is the Pontryagin dual of K . Since the profinite abelian groups are precisely the Pontryagin duals of the torsion abelian groups, one can announce this also in the following form (we call this kind of result a Bridge Theorem):

Theorem 1.1. [30] *Let G be a torsion abelian group and $\phi \in \text{End}(G)$. Then $\text{ent}(\phi) = h_{top}(\hat{\phi})$.*

Later on Peters [21] extended the above definition of algebraic entropy (we denote the Peters' algebraic entropy by h) to automorphisms ϕ of arbitrary abelian groups using, instead of $T_n(\phi, F) = F + \phi(F) + \dots + \phi^{n-1}(F)$, the “negative part of the orbit” $T_n^-(\phi, F) = F + \phi^{-1}(F) + \dots + \phi^{-n+1}(F)$ and instead of finite subgroups F of G just non-empty finite subsets F of G .

Peters proved another Bridge Theorem connecting the topological entropy and the algebraic entropy h by means of the Pontryagin duality:

Theorem 1.2. [21] *Let G be a countable abelian group and $\phi \in \text{Aut}(G)$. Then $h(\phi) = h_{\text{top}}(\widehat{\phi})$.*

Since $T_n^-(\phi, F) = T_n(\phi^{-1}, F)$, the definition by Peters for automorphisms ϕ of abelian groups G can be given also using $T_n(\phi, F)$. The approach using $T_n(\phi, F)$ has the advantage to be applicable also to endomorphisms ϕ (whereas $T_n^-(\phi, F)$ may give rise to an infinite subset of G when ϕ is not injective, so one cannot make recourse to $|T_n^-(\phi, F)|$ to define the algebraic entropy). So we define the algebraic entropy h for endomorphisms ϕ of abelian groups G as follows.

For a non-empty subset F of G and for any positive integer n , the n -th ϕ -trajectory of F is

$$T_n(\phi, F) = F + \phi(F) + \dots + \phi^{n-1}(F),$$

and the ϕ -trajectory of F is $T(\phi, F) = \sum_{n \in \mathbb{N}} \phi^n(F)$. For F finite, let

$$H(\phi, F) = \lim_{n \rightarrow \infty} \frac{\log |T_n(\phi, F)|}{n} \quad (1.3)$$

be the *algebraic entropy* of ϕ with respect to F (this limit exists as proved in Corollary 2.2). The *algebraic entropy* of ϕ is

$$h(\phi) = \sup_{F \in [G]^{<\omega}} H(\phi, F).$$

In particular, for endomorphisms of torsion abelian groups, h coincides with the already defined algebraic entropy ent . More precisely, $\text{ent}(\phi) = h(\phi \upharpoonright_{t(G)})$, where $t(G)$ denotes the torsion subgroup of G .

In Section 2 we prove the basic properties of h (see Lemma 2.7 and Proposition 2.8). These properties are counterparts of the properties of ent . Moreover we give many examples, starting from the fact that the identical homomorphism has zero algebraic entropy. For ent it is obvious that the identical homomorphism has entropy 0, but for h the proof (given in [2]) requires some more effort.

One of our main aims in this paper is to prove the Addition Theorem for the algebraic entropy h , namely, the following

Theorem 1.3 (Addition Theorem). *Let G be an abelian group, $\phi \in \text{End}(G)$, H a ϕ -invariant subgroup of G and $\bar{\phi} : G/H \rightarrow G/H$ the endomorphism induced by ϕ . Then $h(\phi) = h(\phi \upharpoonright_H) + h(\bar{\phi})$.*

Since h coincides with ent for endomorphisms of torsion abelian group, the Addition Theorem for ent proved in [5] for endomorphisms of torsion abelian groups covers the torsion case of Theorem 1.3.

It is convenient to adopt the following notation: for an abelian group G , $\phi \in \text{End}(G)$ and H a ϕ -invariant subgroup of G , we write $\text{AT}_h(G, \phi, H)$ to indicate that the Addition Theorem 1.3 holds for the triple (G, ϕ, H) .

In Sections 2, 3 and 4 we first give some technical results, which permit to reduce the proof of the Addition Theorem 1.3 to appropriate particular cases. In particular, Lemma 3.2 is a reduction to the case of countable abelian groups. Moreover, we consider many properties of the algebraic entropy of endomorphisms of torsion-free abelian groups, still with the aim of proving the Addition Theorem 1.3. For example, for a torsion-free abelian group G and $\phi \in \text{End}(G)$, we see that $h(\phi) = h(\tilde{\phi})$, where $\tilde{\phi} : D(G) \rightarrow D(G)$ is the extension of ϕ to the divisible hull $D(G)$ of G (see Proposition 2.12). This allows to reduce the study of the algebraic entropy of the endomorphisms of torsion-free abelian groups to the case of divisible abelian groups. Moreover, we can reduce to the case of divisible torsion-free abelian group of finite rank. Now the endomorphism ϕ of G can be supposed to be injective (by Proposition 4.5) and then ϕ is also surjective. Finally this particular case of an automorphism of \mathbb{Q}^n can be managed through the Algebraic Yuzvinski Formula:

Theorem 1.4 (Algebraic Yuzvinski Formula). *For $n \in \mathbb{N}_+$ an automorphism ϕ of \mathbb{Q}^n is described by a matrix $A \in GL_n(\mathbb{Q})$. Then*

$$h(\phi) = \log s + \sum_{|\alpha_i| > 1} \log |\alpha_i|, \quad (1.4)$$

where α_i are the eigenvalues of A and s is the least common multiple of the denominators of the coefficients of the (monic) characteristic polynomial of A .

The counterpart of this formula was proved by Yuzvinsky [31] (see also [16]) for the topological entropy (see Theorem 7.4) and it implies (1.4) for h in view of Theorem 1.2. This proof is given in [2].

Let us thickly underline that the proof of the Addition Theorem 1.3 heavily uses the Algebraic Yuzvinski Formula (1.4) (already in the finite-rank torsion-free case). Needless to say, the value of this achievement will be much higher if a purely algebraic proof of the Algebraic Yuzvinski Formula (1.4) would be available.

The proof of the Addition Theorem 1.3 is given in Section 5.

In Section 6 we prove the following Uniqueness Theorem, inspired by the Uniqueness Theorem proved in [5] for ent in the class of torsion abelian groups. It was inspired by the Uniqueness Theorem for the topological entropy by Stoyanov [26].

For any abelian group K the (right) Bernoulli shift $\beta_K : K^{(\mathbb{N})} \rightarrow K^{(\mathbb{N})}$ is defined by

$$(x_0, x_1, x_2, \dots) \mapsto (0, x_0, x_1, \dots).$$

Theorem 1.5 (Uniqueness Theorem). *The algebraic entropy h of the endomorphisms of the abelian groups is characterized as the unique collection $h = \{h_G : G \text{ abelian group}\}$ of functions $h_G : \text{End}(G) \rightarrow \mathbb{R}_+$ such that:*

- (a) h_G is invariant under conjugation for every abelian group G ;
- (b) if $\phi \in \text{End}(G)$ and the group G is a direct limit of ϕ -invariant subgroups $\{G_i : i \in I\}$, then $h_G(\phi) = \sup_{i \in I} h_{G_i}(\phi|_{G_i})$;
- (c) the Addition Theorem holds for h ;
- (d) $h_{K^{(\mathbb{N})}}(\beta_K) = \log |K|$ for any finite abelian group K ;
- (e) the Algebraic Yuzvinski Formula holds for $h_{\mathbb{Q}}$ restricted to the automorphisms of \mathbb{Q} .

Moreover, we see how this result can be deduced by a theorem of Vámos on length functions [19, 27].

In Section 7 we generalize Theorems 1.1 and 1.2 to all endomorphisms of all abelian groups:

Theorem 1.6 (Bridge Theorem). *Let G be an abelian group and $\phi \in \text{End}(G)$. Then $h(\phi) = h_{\text{top}}(\hat{\phi})$.*

To prove this theorem we use its weaker forms proved by Weiss and Peters (see Theorems 1.1 and 1.2 respectively) and we apply the Addition Theorem 1.3 for the algebraic entropy and the Addition Theorem 7.3 for the topological entropy.

In Section 8 we first recall the Mahler measure, which is an important invariant studied in number theory and arithmetic geometry. Moreover we see that the problem of determining the infimum of the positive value of the algebraic entropy is equivalent to the famous Lehmer Problem (see Corollary 8.8).

Notation and terminology

We denote by \mathbb{Z} , \mathbb{N} , \mathbb{N}_+ , \mathbb{Q} and \mathbb{R} respectively the set of integers, the set of natural numbers, the set of positive integers, the set of rationals and the set of reals. For $m \in \mathbb{N}_+$, we use $\mathbb{Z}(m)$ for the finite cyclic group of order m .

Let G be an abelian group. With a slight divergence with the standard use, we denote by $[G]^{<\omega}$ the set of all non-empty finite subsets of G . If H is a subgroup of G , we indicate this by $H \leq G$. The subgroup of torsion elements of G is $t(G)$, while $D(G)$ denotes the divisible hull of G . For a cardinal α we denote by $G^{(\alpha)}$ the direct sum of α many copies of G , that is $\bigoplus_{\alpha} G$.

Moreover, $\text{End}(G)$ is the ring of all endomorphisms of G . We denote by 0_G and id_G respectively the endomorphism of G which is identically 0 and the identity endomorphism of G . Moreover, for $k \in \mathbb{Z}$, let $\mu_k : G \rightarrow G$ be the endomorphism of G defined by $\mu_k(x) = kx$ for every $x \in G$. If $\phi \in \text{End}(G)$, then we denote by $\tilde{\phi} \in \text{End}(D(G))$ the unique extension of ϕ to $D(G)$.

2 Properties and examples

First of all we have to show that, for G an abelian group, $\phi \in \text{End}(G)$ and $F \in [G]^{<\omega}$, the limit defining $H(\phi, F)$ exists. We start proving that $\{\log |T_n(\phi, F)| : n \in \mathbb{N}_+\}$ is a subadditive sequence.

Lemma 2.1. *Let G be an abelian group, $\phi \in \text{End}(G)$ and $F \in [G]^{<\omega}$. For $n \in \mathbb{N}_+$, let $c_n = \log |T_n(\phi, F)|$. Then $c_{n+m} \leq c_n + c_m$ for every $n, m \in \mathbb{N}_+$.*

Proof. By definition

$$\begin{aligned} T_{n+m}(\phi, F) &= F + \phi(F) + \dots + \phi^{n-1}(F) + \phi^n(F) + \dots + \phi^{n+m-1}(F) \\ &= T_n(\phi, F) + \phi^n(T_m(\phi, F)). \end{aligned}$$

Consequently,

$$\begin{aligned} c_{n+m} &= \log |T_{n+m}(\phi, F)| \\ &\leq \log(|T_n(\phi, F)| |T_m(\phi, F)|) \\ &= \log |T_n(\phi, F)| + \log |T_m(\phi, F)| \\ &= c_n + c_m. \end{aligned}$$

□

Corollary 2.2. *Let G be an abelian group, $\phi \in \text{End}(G)$ and $F \in [G]^{<\omega}$. Then the limit $H(\phi, F) = \lim_{n \rightarrow \infty} \frac{\log |T_n(\phi, F)|}{n}$ exists and $H(\phi, F) = \inf_{n \in \mathbb{N}_+} \frac{\log |T_n(\phi, F)|}{n}$.*

Proof. By Lemma 2.1 the sequence $\{c_n : n \in \mathbb{N}_+\}$, where $c_n = \log |T_n(\phi, F)|$, is subadditive. Then the sequence $\{\frac{c_n}{n} : n \in \mathbb{N}_+\}$ has limit and $\lim_{n \rightarrow \infty} \frac{c_n}{n} = \inf_{n \in \mathbb{N}_+} \frac{c_n}{n}$ by a known fact from Calculus, due to Fekete. □

Remark 2.3. Let G be an abelian group and $\phi \in \text{End}(G)$. The function $H(\phi, -)$ is non-decreasing, that is, $H(\phi, F) \leq H(\phi, F')$ for every $F, F' \in [G]^{<\omega}$ with $F \subseteq F'$. Therefore, calculating $h(\phi)$, it is possible to suppose without loss of generality that $0 \in F$. Indeed,

$$h(\phi) = \sup\{H(\phi, F) : F \in [G]^{<\omega}, 0 \in F\}.$$

Intuitively, for any abelian group G , the zero endomorphism 0_G and the identity id_G have zero algebraic entropy. In fact,

Example 2.4. If G is an abelian group, then $h(0_G) = 0$ and $h(id_G) = 0$. While it is clear from the definition that $h(0_G) = 0$, to prove that $h(id_G) = 0$ requires some more effort. In fact, in [2] this is proved showing that the id_G -trajectories of the non-empty finite subsets F of G have polynomial growth; more precisely, for every $F \in [G]^{<\omega}$, there exists $P_F(t) \in \mathbb{Z}[t]$ such that $|T_n(\phi, F)| \leq P_F(n)$ for every $n \in \mathbb{N}_+$ (for id_G one takes $P_F(t) = (t+1)^{|F|}$).

Let G be an abelian group and $\phi \in \text{End}(G)$. A point $x \in G$ is a *periodic* point of ϕ if there exists $n \in \mathbb{N}_+$ such that $\phi^n(x) = x$. Moreover, $x \in G$ is a *quasi-periodic* point of ϕ if there exist $n > m$ in \mathbb{N} such that $\phi^n(x) = \phi^m(x)$. We say that ϕ is *locally (quasi-)periodic* if every $x \in G$ is a (quasi-)periodic point of ϕ . Moreover, ϕ is *periodic* if there exists $n \in \mathbb{N}_+$ such that $\phi^n(x) = x$ for every $x \in G$. Analogously, ϕ is *quasi-periodic* if there exist $n > m$ in \mathbb{N} such that $\phi^n(x) = \phi^m(x)$ for every $x \in G$.

It is possible to prove the following lemma, making use only of the definition of algebraic entropy.

Lemma 2.5. *Let G be an abelian group and $\phi \in \text{End}(G)$. If ϕ is locally quasi-periodic, then $h(\phi) = 0$.*

Proof. Let $F \in [G]^{<\omega}$ and assume without loss of generality that $0 \in F$ (see Remark 2.3). By hypothesis, there exists $m \in \mathbb{N}_+$ such that $\phi^m(F) \subseteq T_m(\phi, F)$. Consequently, $T_n(\phi, F) = T_m(\phi, F)$ for every $n \in \mathbb{N}$, $n \geq m$. Hence $H(\phi, F) = 0$. By the arbitrariness of F , this proves $h(\phi) = 0$. □

In particular, every endomorphism of a finite abelian group has zero algebraic entropy.

Let G be an abelian group and $\phi \in \text{End}(G)$; the *hyperkernel* of ϕ is

$$\ker_\infty \phi = \bigcup_{n \in \mathbb{N}_+} \ker \phi^n.$$

The subgroup $\ker_\infty \phi$ is ϕ -invariant and also invariant for inverse images. Hence the induced endomorphism $\bar{\phi} : G/\ker_\infty \phi \rightarrow G/\ker_\infty \phi$ is injective. Since $\phi \upharpoonright_{\ker_\infty \phi}$ is locally nilpotent for every, and locally nilpotent implies locally quasi-periodic, the following is an immediate consequence of Lemma 2.5.

Corollary 2.6. *Let G be an abelian group and $\phi \in \text{End}(G)$. Then $h(\phi \upharpoonright_{\ker_\infty \phi}) = 0$.*

In the next lemma we show that h is monotone under taking restriction to invariant subgroups and under taking induced endomorphisms on quotients over invariant subgroups.

Lemma 2.7. *Let G be an abelian group, $\phi \in \text{End}(G)$, H a ϕ -invariant subgroup of G and $\bar{\phi} : G/H \rightarrow G/H$ the endomorphism induced by ϕ . Then $h(\phi) \geq \max\{h(\phi|_H), h(\bar{\phi})\}$.*

Proof. For every $F \in [H]^{<\omega}$, obviously $H(\phi|_H, F) = H(\phi, F)$, so $H(\phi|_H, F) \leq h(\phi)$. Hence $h(\phi|_H) \leq h(\phi)$.

Now assume that $F \in [G/H]^{<\omega}$ and $F = \pi(F_0)$ for some $F_0 \in [G]^{<\omega}$, where $\pi : G \rightarrow G/H$ is the canonical projection. Then $\pi(T_n(\phi, F_0)) = T_n(\bar{\phi}, F)$ for every $n \in \mathbb{N}_+$. Therefore, $H(\phi, F_0) \geq H(\bar{\phi}, F)$ and by the arbitrariness of F this proves $h(\phi) \geq h(\bar{\phi})$. \square

The properties in the next proposition are the basic ones for h . They are the typical properties of the known entropies. Indeed, similar properties holds for the algebraic entropy ent , for ent_i , and also for the topological entropy (see Fact 7.1), which gave the inspiration. In the case of the algebraic entropy h they were proved in [21] for automorphisms, and we extend them for endomorphisms.

Proposition 2.8. *Let G be an abelian group and $\phi \in \text{End}(G)$.*

- (a) *If H is another abelian group, $\eta \in \text{End}(H)$ and ϕ and η are conjugated (i.e., there exists an isomorphism $\xi : G \rightarrow H$ such that $\eta = \xi\phi\xi^{-1}$), then $h(\phi) = h(\eta)$.*
- (b) *For every $k \in \mathbb{N}_+$, $h(\phi^k) = kh(\phi)$. If ϕ is an automorphism, then $h(\phi^k) = |k|h(\phi)$ for every $k \in \mathbb{Z}$.*
- (c) *If G is a direct limit of ϕ -invariant subgroups $\{G_i : i \in I\}$, then $h(\phi) = \sup_{i \in I} h(\phi|_{G_i})$.*
- (d) *If $G = G_1 \times G_2$ and $\phi = \phi_1 \times \phi_2$ with $\phi_i \in \text{End}(G_i)$, $i = 1, 2$, then $h(\phi_1 \times \phi_2) = h(\phi_1) + h(\phi_2)$.*

Proof. (a) For $F \in [G]^{<\omega}$ and $n \in \mathbb{N}_+$, $T_n(\eta, \xi(F)) = \xi(F) + \xi(\phi(F)) + \dots + \xi(\phi^{n-1}(F))$. Since ξ is an isomorphism, $|T_n(\phi, F)| = |T_n(\eta, \xi(F))|$, and so $H(\phi, F) = H(\eta, \xi(F))$. This proves that $h(\phi) = h(\eta)$.

(b) Fix $k \in \mathbb{N}_+$. First we prove the inequality $h(\phi^k) \leq kh(\phi)$. Let $F \in [G]^{<\omega}$, assuming without loss of generality that $0 \in F$ (see Remark 2.3), and let $n \in \mathbb{N}_+$. Then $T_n(\phi^k, F) \subseteq T_{kn-k+1}(\phi, F)$ and so

$$\begin{aligned} H(\phi^k, F) &= \lim_{n \rightarrow \infty} \frac{\log |T_n(\phi^k, F)|}{n} \\ &\leq \lim_{n \rightarrow \infty} \frac{\log |T_{kn-k+1}(\phi, F)|}{n} \\ &= \lim_{n \rightarrow \infty} \frac{\log |T_{kn-k+1}(\phi, F)|}{kn-k+1} \cdot \lim_{n \rightarrow \infty} \frac{kn-k+1}{n} \\ &= k \lim_{n \rightarrow \infty} \frac{\log |T_{kn-n+1}(\phi, F)|}{kn-k+1} \\ &= kH(\phi, F). \end{aligned}$$

Therefore, $h(\phi^k) \leq kh(\phi)$.

To prove the converse inequality $h(\phi^k) \geq kh(\phi)$, let $F \in [G]^{<\omega}$ and $n \in \mathbb{N}_+$. Let $F_1 = T_k(\phi, F)$ and note that $T_n(\phi^k, F_1) = T_{kn}(\phi, F)$. Then

$$\frac{1}{k}H(\phi^k, F_1) = \lim_{n \rightarrow \infty} \frac{\log |T_n(\phi^k, F_1)|}{kn} = \lim_{n \rightarrow \infty} \frac{\log |T_{kn}(\phi, F)|}{kn} = H(\phi, F).$$

We can conclude that $h(\phi^k) \geq kh(\phi)$.

Now assume that ϕ is an automorphism. It suffices to prove that $h(\phi^{-1}) = h(\phi)$. Let $F \in [G]^{<\omega}$ and $n \in \mathbb{N}_+$. Note that $T_n(\phi^{-1}, F) = \phi^{-n+1}T_n(\phi, F)$; in particular, $|T_n(\phi^{-1}, F)| = |T_n(\phi, F)|$, as ϕ is an automorphism. This yields $H(\phi^{-1}, F) = H(\phi, F)$, hence $h(\phi^{-1}) = h(\phi)$.

(c) By Lemma 2.7, $h(\phi) \geq h(\phi|_{G_i})$ for every $i \in I$ and so $h(\phi) \geq \sup_{i \in I} h(\phi|_{G_i})$.

To prove the converse inequality let $F \in [G]^{<\omega}$. Since $G = \varinjlim \{G_i : i \in I\}$ and $\{G_i : i \in I\}$ is a directed family, there exists $j \in I$ such that $F \subseteq G_j$. Then $H(\phi, F) = \overline{H(\phi|_{G_j}, F)} \leq h(\phi|_{G_j})$. This proves that $h(\phi) \leq \sup_{i \in I} h(\phi|_{G_i})$.

(d) Fix $F_i \in [G_i]^{<\omega}$, for $i = 1, 2$. Then

$$T_n(\phi, F_1 \times F_2) = T_n(\phi_1, F_1) \times T_n(\phi_2, F_2),$$

and so

$$H(\phi, F_1 \times F_2) = H(\phi_1, F_1) + H(\phi_2, F_2). \quad (2.1)$$

Consequently, $h(\phi) \geq h(\phi_1) + h(\phi_2)$. Since every $F \in [G]^{<\omega}$ is contained in some $F_1 \times F_2$, for $F_i \in [G_i]^{<\omega}$, $i = 1, 2$, and so $H(\phi, F) \leq H(\phi, F_1 \times F_2)$, hence (2.1) proves also that $h(\phi) \leq h(\phi_1) + h(\phi_2)$. \square

The next is a direct consequence of Proposition 2.8(b).

Corollary 2.9. *Let G be an abelian group and $\phi \in \text{End}(G)$. Then:*

- (a) $h(\phi) = 0$ if and only if $h(\phi^k) = 0$ for some $k \in \mathbb{N}_+$, and
- (b) $h(\phi) = \infty$ if and only if $h(\phi^k) = \infty$ for some $k \in \mathbb{N}_+$.

The following is a fundamental example in the theory of algebraic entropy. Indeed, the value of h on the right Bernoulli shift is one of the conditions that give uniqueness of h , as we will show in Section 6.

Example 2.10. It is proved in [5] that $h(\beta_{\mathbb{Z}(p)}) = \text{ent}(\beta_{\mathbb{Z}(p)}) = \log p$, for every prime p . Equivalently, $h(\beta_K) = \text{ent}(\beta_K) = \log |K|$ for every finite abelian group K .

Then $h(\beta_{\mathbb{Z}}) = \infty$. Indeed, let $G = \mathbb{Z}^{(\mathbb{N})}$; for every prime p , the subgroup pG of G is $\beta_{\mathbb{Z}}$ -invariant, so induces an endomorphism $\overline{\beta_{\mathbb{Z}}} : G/pG \rightarrow G/pG$. Since $G/pG \cong \mathbb{Z}(p)^{(\mathbb{N})}$, and $\overline{\beta_{\mathbb{Z}}}$ is conjugated to $\beta_{\mathbb{Z}(p)}$ through this isomorphism, $h(\overline{\beta_{\mathbb{Z}}}) = h(\beta_{\mathbb{Z}(p)}) = \log p$ by Proposition 2.8(a). Therefore $h(\beta_{\mathbb{Z}}) \geq \log p$ for every prime p by Lemma 2.7, and so $h(\beta_{\mathbb{Z}}) = \infty$.

Assume that K is an infinite abelian group. If K is non-torsion, then K contains a subgroup $C \cong \mathbb{Z}$, so $K^{(\mathbb{N})}$ contains the β_K -invariant subgroup $C^{(\mathbb{N})}$ isomorphic to $\mathbb{Z}^{(\mathbb{N})}$. Hence, by Lemma 2.7, Proposition 2.8(a) and the previous part of this example, $h(\beta_K) \geq h(\beta_C) = h(\beta_{\mathbb{Z}}) = \infty$. If K is torsion, then K contains arbitrarily large finite subgroups H . Consequently, $K^{(\mathbb{N})}$ contains the β_K -invariant subgroup $H^{(\mathbb{N})}$. By Lemma 2.7 and the first part of this example, $h(\beta_K) \geq h(\beta_H) = \log |H|$ for every H . So $h(\beta_K) = \infty$.

Hence, for any abelian group K , $h(\beta_K) = \log |K|$, with the usual convention that $\log |K| = \infty$, if $|K|$ is infinite.

Lemma 2.11. *Let G be a torsion-free abelian group and $\phi \in \text{End}(G)$. If $G = V(\phi, g)$ for some $g \in G$ and $h(\phi) < \infty$, then G has finite rank.*

Proof. Assume that $r(G)$ is infinite. This entails $\langle \phi^m(g) \rangle \cap T_m(\phi, g) = 0$ for every $m \in \mathbb{N}_+$, so $T_m(\phi, g) = \bigoplus_{k=1}^{m-1} \langle \phi^k(x) \rangle$ and $G = \bigoplus_{k \in \mathbb{N}} \langle \phi^k(x) \rangle \cong \mathbb{Z}^{(\mathbb{N})}$. Since ϕ is conjugated to $\beta_{\mathbb{Z}}$ through this isomorphism, by Proposition 2.8(a) and Example 2.10 we conclude that $h(\phi) = h(\beta_{\mathbb{Z}}) = \infty$. \square

The next result reduces the computation of the algebraic entropy of endomorphisms of torsion-free abelian groups to the case of endomorphisms of divisible abelian groups.

Proposition 2.12. *Let G be a torsion-free abelian group and $\phi \in \text{End}(G)$. Then $h(\phi) = h(\tilde{\phi})$.*

Proof. It is obvious that $h(\phi) \leq h(\tilde{\phi})$ by Lemma 2.7.

Let $F \in [D(G)]^{<\omega}$. There exists $m \in \mathbb{N}_+$ such that $mF \subseteq G$. The automorphism μ_m of $D(G)$ commutes with $\tilde{\phi}$. Hence, for $n \in \mathbb{N}_+$, $T_n(\phi, mF) = T_n(\phi, \mu_m(F)) = \mu_m(T_n(\tilde{\phi}, F))$; in particular, $|T_n(\phi, mF)| = |T_n(\tilde{\phi}, F)|$. Therefore, $H(\phi, mF) = H(\tilde{\phi}, F)$. By the arbitrariness of F , this gives $h(\phi) \geq h(\tilde{\phi})$. \square

The following example shows that Proposition 2.12 may fail if G is not the torsion-free.

Example 2.13. Let $G = \mathbb{Z}(2)^{(\mathbb{N})}$. Then $h(\beta_{\mathbb{Z}(2)}) = \text{ent}(\beta_{\mathbb{Z}(2)}) = \log 2$. For $D(G) = \mathbb{Z}(2^\infty)^{(\mathbb{N})}$ instead, $\widetilde{\beta_{\mathbb{Z}(2)}} = \beta_{\mathbb{Z}(2^\infty)}$ has $h(\beta_{\mathbb{Z}(2^\infty)}) = \text{ent}(\beta_{\mathbb{Z}(2^\infty)}) = \infty$ (see Example 2.10).

Recall that, if G is a torsion-free abelian group, then a subgroup H of G is essential if and only if for every $x \in G \setminus \{0\}$ there exists $k \in \mathbb{Z}$ such that $kx \in H \setminus \{0\}$.

Corollary 2.14. *Let G be a torsion-free abelian group, $\phi \in \text{End}(G)$, H a ϕ -invariant subgroup of G and $\bar{\phi} : G/H \rightarrow G/H$ the endomorphism induced by ϕ . Then the purification H_* of H in G is ϕ -invariant and $h(\phi \upharpoonright_H) = h(\bar{\phi} \upharpoonright_{H_*})$. Consequently, if H is an essential subgroup of G , then*

- (a) $h(\phi) = h(\phi \upharpoonright_H)$;
- (b) $h(\phi) < \infty$ implies $h(\bar{\phi}) = 0$.

Proof. For the first assertion see [24, Lemma 3.3(a)]. Consider the divisible hull $D(H)$ of H . We can assume without loss of generality that H_* is a subgroup of $D(H)$. Let $\tilde{\phi} : D(H) \rightarrow D(H)$ denote the common (unique) extension of $\phi \upharpoonright_H$ and $\tilde{\phi} \upharpoonright_{H_*}$. Proposition 2.12 applies to the pairs $D(H), H$ and $D(H), H_*$, giving $h(\phi \upharpoonright_H) = h(\tilde{\phi} \upharpoonright_{D(H)}) = h(\tilde{\phi} \upharpoonright_{H_*})$.

(a) follows from the first assertion, since $G = H_*$.

(b) Fix a prime p . Since G/H is torsion, it suffices to see that $h(\bar{\phi} \upharpoonright_{(G/H)[p]}) = 0$ [5, Proposition 1.18]. To this end we have to show that every $\bar{x} \in (G/H)[p]$ has finite trajectory under $\bar{\phi}$. By Lemma 2.11 $V(\phi, x) \leq G$ has finite rank, say $n \in \mathbb{N}$. Then there exist $k_i \in \mathbb{Z}$, $i = 0, \dots, n$, such that $\sum_{i=0}^n k_i \phi^i(x) = 0$. Since G is torsion-free, we can assume without loss of generality that at least one of these coefficient is not divisible by p . Now projecting in G/H we conclude that $\sum_{i=0}^n k_i \bar{\phi}^i(\bar{x}) = 0$ is a non-trivial linear combination in $(G/H)[p]$. Hence $\bar{x} \in (G/H)[p]$ has finite trajectory under $\bar{\phi}$. \square

In the following example we calculate the algebraic entropy of the endomorphisms of \mathbb{Z} and \mathbb{Q} , and of the multiplications of torsion-free abelian groups. In particular, item (a) immediately shows a difference with the torsion case. For an abelian group G and $\phi \in \text{End}(G)$, ϕ is *integral* if there exists $f(t) \in \mathbb{Z}[t] \setminus \{0\}$ such that $f(\phi) = 0$. According to [5, Lemma 2.2], if G is torsion, ϕ integral implies $\text{ent}(\phi) = 0$. On the other hand, for $k > 1$ the endomorphism $\mu_k : \mathbb{Z} \rightarrow \mathbb{Z}$ in (a) of the next example is integral over \mathbb{Z} (as $\mu_k(x) - kx = 0$ for all $x \in \mathbb{Z}$), nevertheless, $h(\mu_k) = \log k > 0$.

Example 2.15. (a) For every $k \in \mathbb{N}_+$, $\mu_k : \mathbb{Z} \rightarrow \mathbb{Z}$ has $h(\mu_k) = \log k$.

For $k = 1$ this follows from Example 2.4. Assume $k > 1$ and let $F_0 = \{0, 1, \dots, k-1\} \in [\mathbb{Z}]^{<\omega}$, $n \in \mathbb{N}_+$. Every $m \in \mathbb{N}$ with $m < k^n$ can be uniquely written in the form $m = f_0 + f_1 k + \dots + f_{n-1} k^{n-1}$ with all $f_i \in F_0$. Then $T_n(\mu_k, F_0) = \{m \in \mathbb{N} : m < k^n\}$, and so $|T_n(\mu_k, F_0)| = k^n$. Consequently $H(\mu_k, F_0) = \log k$ and this yields $h(\mu_k) \geq \log k$.

To prove the converse inequality $h(\mu_k) \leq \log k$, fix $m \in \mathbb{N}_+$ and let $F_m = \{0, \pm 1, \pm 2, \dots, \pm m\} \in [\mathbb{Z}]^{<\omega}$. Then $|x| \leq mk^n$ for every $x \in T_n(\mu_k, F_m)$. So $|T_n(\mu_k, F_m)| \leq 3mk^n$, hence $H(\mu_k, F_m) \leq \log k$. Since each $F \in [\mathbb{Z}]^{<\omega}$ is contained in some F_m for some $m \in \mathbb{N}_+$, we obtain $h(\mu_k) \leq \log k$.

(b) Let $\phi \in \text{End}(\mathbb{Q})$. Then $\phi = \mu_r$, with $r \in \mathbb{Q}$. If $r = 0, \pm 1$, then $h(\phi) = 0$ by Example 2.4. Applying Proposition 2.8(b), we may assume that $r > 1$. Let $r = \frac{a}{b}$, where $(a, b) = 1$. Then $h(\phi) = \log a$.

To prove that $h(\phi) \geq \log a$, take $F_0 = \{0, 1, \dots, a-1\}$. Let us check that all sums $f_0 + f_1 \frac{a}{b} + \dots + f_{n-1} \frac{a^{n-1}}{b^{n-1}}$, with $f_i \in F_0$, are pairwise distinct. Indeed, assume that

$$f_0 + f_1 \frac{a}{b} + \dots + f_{n-1} \frac{a^{n-1}}{b^{n-1}} = f'_0 + f'_1 \frac{a}{b} + \dots + f'_{n-1} \frac{a^{n-1}}{b^{n-1}} \quad (2.2)$$

for some $f_i, f'_i \in F_0$. Then $f_0 b^{n-1} + f_1 a b^{n-2} + \dots + f_{n-1} a^{n-1} = f'_0 b^{n-1} + f'_1 a b^{n-2} + \dots + f'_{n-1} a^{n-1}$, so that a divides $f_0 b^{n-1} - f'_0 b^{n-1} = b^{n-1}(f_0 - f'_0)$. As $(a, b) = 1$, we conclude that a divides $f_0 - f'_0$ and this obviously entails $f_0 = f'_0$. Consequently, (2.2) gives $f_1 + f_2(\frac{a}{b}) + \dots + f_{n-1}(\frac{a}{b})^{n-2} = f'_1 + f'_2(\frac{a}{b}) + \dots + f'_{n-1}(\frac{a}{b})^{n-2}$. Now an obvious induction argument applies. Therefore, this shows that $|T_n(\phi, F_0)| = a^n$, and so $H_n(\phi, F_0) = n \log a$. Thus $h(\phi) \geq \log a$.

To prove the inequality $h(\phi) \leq \log a$, note that the subgroup H of \mathbb{Q} formed by all fractions having as denominators powers of b (i.e., the subring of \mathbb{Q} generated by $\frac{1}{b}$), is ϕ -invariant.

Since $H \supseteq \mathbb{Z}$, H is essential in \mathbb{Q} , and so $h(\phi) = h(\phi \upharpoonright_H)$ by Corollary 2.14(a). Now for any $m \in \mathbb{N}_+$ consider $F_m = \{\pm \frac{r}{b^m} : 0 \leq r \leq mb^m\}$. So $F_m = \langle \frac{1}{b^m} \rangle \cap [-m, m]$, where the interval $[-m, m]$ is taken in H . Let us observe that $\phi^k(F_m) \subseteq \langle \frac{1}{b^{m+k}} \rangle \cap [-m \frac{a^k}{b^k}, m \frac{a^k}{b^k}]$, consequently

$$T_n(\phi, F_m) \subseteq \underbrace{M + \dots + M}_n$$

where $M = \langle \frac{1}{b^{m+n-1}} \rangle \cap [-m \frac{a^{n-1}}{b^{n-1}}, m \frac{a^{n-1}}{b^{n-1}}]$. Therefore, $|T_n(\phi, F_m)| \leq 2nb^{m+n-1}m \frac{a^{n-1}}{b^{n-1}}$. Hence,

$$\log |T_n(\phi, F_m)| \leq \log 2n + (m+n-1) \log b + \log m + (n-1)(\log a - \log b) = \log 2n + m \log b + (n-1) \log a.$$

Thus $H(\phi, F_m) \leq \log a$. Since every $F \in [H]^{<\omega}$ is contained in F_m for some $m \in \mathbb{N}_+$, this proves that $h(\phi) = h(\phi \upharpoonright_H) \leq \log a$.

(c) Let $n \in \mathbb{N}_+$.

- (i) For $k \in \mathbb{N}_+$ and $\mu_k : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$, $h(\mu_k) = n \log k$.
- (ii) For $r = \frac{a}{b} \in \mathbb{Q}$ with $a > b > 0$, and $\mu_r : \mathbb{Q}^n \rightarrow \mathbb{Q}^n$, $h(\mu_r) = n \log a$.

To verify (i) and (ii), it suffices to apply Proposition 2.8(d) and (a) and (b) respectively.

- (d) Let G be a torsion-free abelian group and consider $\mu_k : G \rightarrow G$ for some $k \in \mathbb{N}_+$. Then

$$h(\mu_k) = \begin{cases} r(G) \log k & \text{if } r(G) \text{ is finite,} \\ \infty & \text{if } r(G) \text{ is infinite.} \end{cases}$$

Let $\alpha = r(G)$. Then $D(G) \cong \mathbb{Q}^{(\alpha)}$, with $\widetilde{\mu_k}$ conjugated to the multiplication by k $\mu_k^{\mathbb{Q}} : \mathbb{Q}^{(\alpha)} \rightarrow \mathbb{Q}^{(\alpha)}$, and G has a subgroup H isomorphic to $\mathbb{Z}^{(\alpha)}$, which is μ_k -invariant, with $\mu_k \upharpoonright_H$ conjugated to the multiplication by k $\mu_k^{\mathbb{Z}} : \mathbb{Z}^{(\alpha)} \rightarrow \mathbb{Z}^{(\alpha)}$. Assume that $\alpha \in \mathbb{N}$. By (ii) of (c) and Proposition 2.8(a), $h(\widetilde{\mu_k}) = h(\mu_k^{\mathbb{Q}}) = \alpha \log k$. By (i) of (c) and Proposition 2.8(a), $h(\mu_k \upharpoonright_H) = h(\mu_k^{\mathbb{Z}}) = \alpha \log k$. Then $h(\mu_k) = \alpha \log k$ by Lemma 2.7. If α is infinite, by Lemma 2.7 and in view of the finite case, $h(\mu_k \upharpoonright_H) = h(\mu_k^{\mathbb{Z}}) > n \log k$ for every $n \in \mathbb{N}$. Hence $h(\mu_k \upharpoonright_H) = \infty$ and so $h(\mu_k) = \infty$ by Lemma 2.7.

In item (b) of the above example we have given the explicit computation of the entropy of $\mu_r : \mathbb{Q} \rightarrow \mathbb{Q}$, with $r = \frac{a}{b} > 1$ and $(a, b) = 1$. One can also apply the Algebraic Yuzvinski Formula (1.4); indeed, the unique eigenvalue of μ_r is $\frac{a}{b} > 1$, and so (1.4) gives $h(\mu_r) = \log a$. This formula was given by Abramov for the topological entropy of the automorphisms of $\widehat{\mathbb{Q}}$.

Example 2.16. Fix $k \in \mathbb{Z}$ and consider the automorphism $\phi : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ defined by $\phi(x, y) = (x + ky, y)$ for all $(x, y) \in \mathbb{Z}^2$. Then $h(\phi) = 0$.

Let $m \in \mathbb{N}_+$ and $F_m = \{0, \pm 1, \pm 2, \dots, \pm m\} \times \{0, \pm 1, \pm 2, \dots, \pm m\} \in [\mathbb{Z}^2]^{<\omega}$. Every $F \in [\mathbb{Z}^2]^{<\omega}$ is contained in F_m for some $m \in \mathbb{N}_+$. Therefore it suffices to show that $H(\phi, F_m) = 0$. One can prove by induction that, for every $n \in \mathbb{N}_+$, $T_n(\phi, F_m)$ is contained in a parallelogram with sides $2nm$ and $nm(2 + nk - k)$, so $|T_n(\phi, F_m)| \leq 2n^2m^2(2 + nk - k)$. Thus $H(\phi, F_m) = 0$.

Let G be an abelian group and $\phi \in \text{End}(G)$. For $F \subseteq G$, the trajectory $T(\phi, F)$ needs not be a subgroup of G . So let

$$V(\phi, F) = \langle \phi^n(F) : n \in \mathbb{N} \rangle = \langle T(\phi, F) \rangle.$$

This is the smallest ϕ -invariant subgroup containing $T(\phi, F)$ (and so also F). If $F = \{g\}$ we denote $V(\phi, \{g\})$ simply by $V(\phi, g)$. For $F \in [G]^{<\omega}$, $V(\phi, F) = \sum_{g \in F} V(\phi, g)$. Note that $V(\phi, F)$ is the $\mathbb{Z}[t]$ -module generated by F . Indeed, G has structure of $\mathbb{Z}[t]$ -module given by ϕ : the multiplication by t is defined by $tx = \phi(x)$ for every $x \in G$. This will be discussed with more details in Section 6.

Lemma 2.17. *Let G be an abelian group and $\phi \in \text{End}(G)$. Assume that $G = V(\phi, F)$ for some $F \in [G]^{<\omega}$, i.e., G is finitely generated (by F) as a $\mathbb{Z}[t]$ -module.*

- (a) *Every subgroup and every quotient of G is finitely generated as a $\mathbb{Z}[t]$ -module.*
- (b) *If ϕ is locally periodic, then ϕ is periodic.*
- (c) *There exists $m \in \mathbb{N}_+$ such that $mt(G) = 0$.*

Proof. (a) It follows from the fact that $\mathbb{Z}[t]$ is Noetherian.

- (b) There exists $m \in \mathbb{N}_+$ such that $\phi^m \upharpoonright_F = \text{id}_F$. Since F generates G , $\phi^m = \text{id}_G$.

- (c) By (a) $t(G)$ is finitely generated as a $\mathbb{Z}[t]$ -module, that is $t(G) = V(\phi, F')$ for some $F' \in [G]^{<\omega}$. Since F' is finite, there exists $m \in \mathbb{N}_+$ such that $mF' = 0$. Then $mt(G) = 0$. \square

Lemma 2.18. *Let G be an abelian group and $\phi \in \text{End}(G)$. Then $G = \varinjlim \{V(\phi, F) : F \in [G]^{<\omega}\}$, and so $h(\phi) = \sup\{h(\phi \upharpoonright_{V(\phi, F)}) : F \in [G]^{<\omega}\}$.*

Proof. The family $\{V(\phi, F) : F \in [G]^{<\omega}\}$ is a direct system; indeed, for every $F_1, F_2 \in [G]^{<\omega}$, $V(\phi, F_1) \cup V(\phi, F_2) \subseteq V(\phi, F_1 \cup F_2)$. That $h(\phi) = \sup\{h(\phi \upharpoonright_{V(\phi, F)}) : F \in [G]^{<\omega}\}$ follows from Proposition 2.8(c). \square

Since h is defined “locally”, in some sense this lemma permits to reduce to the case $G = V(\phi, F)$ for some $F \in [G]^{<\omega}$, that is, G is finitely generated (by F) as a $\mathbb{Z}[t]$ -module. By Lemma 2.17(a) every subgroup and every quotient of G is finitely generated as a $\mathbb{Z}[t]$ -module.

Proposition 2.19. *Let G be a countable abelian group, $\phi \in \text{End}(G)$ and H a ϕ -invariant subgroup of G . Then there exists a family $\{L_n : n \in \mathbb{N}\} \subseteq [G]^{<\omega}$, such that:*

$$\begin{aligned} h(\phi) &= \lim_{n \rightarrow \infty} h(\phi \upharpoonright_{V(\phi, L_n)}), \\ h(\phi \upharpoonright_H) &= \lim_{n \rightarrow \infty} h(\phi \upharpoonright_{H \cap V(\phi, L_n)}), \text{ and} \\ h(\bar{\phi}) &= \lim_{n \rightarrow \infty} h(\bar{\phi} \upharpoonright_{\pi(L_n)}). \end{aligned}$$

Proof. We prove that, whenever G is a countable and $\phi \in \text{End}(G)$,

- (i) there exists $\{F_n : n \in \mathbb{N}\} \subseteq [G]^{<\omega}$ such that G is increasing union of the $V(\phi, F_n)$;
- (ii) consequently, $h(\phi) = \lim_{n \rightarrow \infty} h(\phi \upharpoonright_{V(\phi, F_n)})$.

Let $G = \{g_n : n \in \mathbb{N}\}$, and for every $n \in \mathbb{N}$ let $F_n = \{g_0, \dots, g_n\}$. Then G is increasing union of the F_n and consequently of the $V(\phi, F_n)$. By Proposition 2.8(d) $h(\phi) = \sup_{n \in \mathbb{N}} h(\phi \upharpoonright_{V(\phi, F_n)}) = \lim_{n \rightarrow \infty} h(\phi \upharpoonright_{V(\phi, F_n)})$, as $\{h(\phi \upharpoonright_{V(\phi, F_n)}) : n \in \mathbb{N}\}$ is a non-decreasing sequence.

By (i) and (ii) applied to H and $\phi \upharpoonright_H$ there exists $\{F'_n : n \in \mathbb{N}\} \subseteq [H]^{<\omega}$, such that $H = \bigcup_{n \in \mathbb{N}} V(\phi \upharpoonright_H, F'_n)$, where this is an increasing union, and $h(\phi) = \lim_{n \rightarrow \infty} h(\phi \upharpoonright_{V(\phi \upharpoonright_H, F'_n)})$. Let $L_n = F_n \cup F'_n$. Then $\{L_n : n \in \mathbb{N}\} \subseteq [G]^{<\omega}$ is such that $G = \bigcup_{n \in \mathbb{N}} V(\phi, L_n)$, where this is an increasing union. By (ii)

$$h(\phi) = \lim_{n \rightarrow \infty} h(\phi \upharpoonright_{V(\phi, L_n)}).$$

Since $V(\phi \upharpoonright_H, F'_n) \subseteq H \cap V(\phi, L_n)$ for every $n \in \mathbb{N}$, $H = \bigcup_{n \in \mathbb{N}} (H \cap V(\phi, L_n))$, where this is an increasing union, and so by (ii)

$$h(\phi \upharpoonright_H) = \lim_{n \rightarrow \infty} h(\phi \upharpoonright_{H \cap V(\phi, L_n)}).$$

For $\pi : G \rightarrow G/H$ the canonical projection, $G/H = \bigcup_{n \in \mathbb{N}} \pi(V(\phi, L_n)) = \bigcup_{n \in \mathbb{N}} V(\bar{\phi}, \pi(L_n))$, where $\{\pi(L_n) : n \in \mathbb{N}\} \subseteq [G/H]^{<\omega}$ and this is an increasing union. By (ii) applied to G/H and $\bar{\phi}$,

$$h(\bar{\phi}) = \lim_{n \rightarrow \infty} h(\bar{\phi} \upharpoonright_{\pi(L_n)}),$$

and this concludes the proof. □

3 The club of supports, the skew products and various relations for the Addition Theorem

Definition 3.1. *Let G be an abelian group and $\phi \in \text{End}(G)$. An entropy support of (G, ϕ) is a countable ϕ -invariant subgroup S of G such that $h(\phi \upharpoonright_S) = h(\phi)$.*

Clearly, every countable ϕ -invariant subgroup of G containing an entropy support, will have the same property, that is, such a subgroup is not uniquely determined. In the sequel we denote by $\mathcal{S}(G, \phi)$ the family of all entropy supports of the pair (G, ϕ) .

The next lemma shows that $\mathcal{S}(G, \phi)$ is not empty, i.e., there exists at least one (and then infinitely many) of such subgroups.

Lemma 3.2. *Let G be an abelian group and $\phi \in \text{End}(G)$. Then there exists a entropy support of (G, ϕ) .*

Proof. Since $h(\phi) = \sup_{F \in [G]^{<\omega}} H(\phi, F)$ is in $\mathbb{R}_+ \cup \{\infty\}$, there exists a family $\{F_n : n \in \mathbb{N}\} \subseteq [G]^{<\omega}$ such that $h(\phi) = \sup_{n \in \mathbb{N}} H(\phi, F_n)$. Let $S = V(\phi, \bigcup_{n \in \mathbb{N}} F_n)$, which is a countable ϕ -invariant subgroup of G , such that $h(\phi) = \sup_{n \in \mathbb{N}} H(\phi, F_n) = \sup_{n \in \mathbb{N}} H(\phi \upharpoonright_S, F_n) = h(\phi \upharpoonright_S)$. □

The family $\mathcal{S}(G, \phi)$ is a club. Let us recall that a family \mathcal{C} of countable subsets of an infinite set X is a *club* (for closed unbounded) if it is closed for countable increasing unions and if every countable subset of X is contained in an element of \mathcal{C} .

The following is a first reduction for the proof of the Addition Theorem 1.3 to countable abelian groups.

Proposition 3.3. *Assume that $\text{AT}_h(G, \phi, H)$ holds for every countable abelian group G , $\phi \in \text{End}(G)$ and H a ϕ -invariant subgroup of G . Then $\text{AT}_h(G, \phi, H)$ holds for every abelian group G , $\phi \in \text{End}(G)$ and H a ϕ -invariant subgroup of G .*

Proof. Let G be an abelian group, $\phi \in \text{End}(G)$, H a ϕ -invariant subgroup of G , $\bar{\phi} : G/H \rightarrow G/H$ the endomorphism induced by ϕ and $\pi : G \rightarrow G/H$ be the canonical projection. Let $S \in \mathcal{S}(G, \phi)$, $S_H \in \mathcal{S}(H, \phi \upharpoonright_H)$ and $\bar{S} \in \mathcal{S}(G/H, \bar{\phi})$. We can assume without loss of generality that $S \supseteq S_H$ and $\pi(S) \supseteq \bar{S}$. Then $S'_H = \ker \pi \upharpoonright_S = S \cap H \supseteq S_H$ and $S'_H \in \mathcal{S}(H, \phi \upharpoonright_H)$. Consequently, S'_H is a ϕ -invariant subgroup of S such that $S/S'_H \cong \pi(S)$. Let $\bar{\phi} \upharpoonright_S : S/S'_H \rightarrow S/S'_H$ be the endomorphism induced by $\phi \upharpoonright_S$, which is conjugated to $\bar{\phi} \upharpoonright_{\pi(S)}$. By hypothesis, and by Proposition 2.8(a),

$$h(\phi \upharpoonright_S) = h(\phi), \quad h(\bar{\phi} \upharpoonright_{\bar{S}}) = h(\bar{\phi}) \quad \text{and} \quad h(\phi \upharpoonright_{S'_H}) = h(\phi \upharpoonright_H). \quad (3.1)$$

Since S is countable, by hypothesis $\text{AT}_h(S, \phi \upharpoonright_S, S'_H)$ holds, and hence (3.1) implies that $\text{AT}_h(G, \phi, H)$ holds as well. \square

Let K, H be abelian groups and $\phi_1 \in \text{End}(K)$, $\phi_2 \in \text{End}(H)$. The direct product $\pi = \phi_1 \times \phi_2 : K \times H \rightarrow K \times H$ is defined by $\pi(x, y) = (\phi_1(x), \phi_2(y))$ for every pair $(x, y) \in K \times H$. For an homomorphism $s : K \rightarrow H$, the *skew product* of ϕ_1 and ϕ_2 via s is $\phi : K \times H \rightarrow K \times H$ defined by

$$\phi(x, y) = (\phi_1(x), \phi_2(y) + s(x)) \quad \text{for every } (x, y) \in K \times H. \quad (3.2)$$

We say that the homomorphism s is *associated* to the skew product ϕ .

Clearly, $H = 0 \times H$ is a ϕ -invariant subgroup of $K \times H$ and the induced endomorphism of $K \cong (K \times H)/H$ is precisely ϕ_1 . When $s = 0$ one obtains the usual direct product endomorphism $\phi = \phi_1 \times \phi_2$. Let us see that the skew products arise precisely in such a circumstance:

Remark 3.4. If G is an abelian group and $\phi \in \text{End}(G)$, suppose to have a ϕ -invariant subgroup H of G that splits as a direct summand, that is $G = K \times H$. Let us see that ϕ is a skew product. Indeed, let $\iota : G/H \rightarrow K$ be the natural isomorphism and let $\bar{\phi} : G/H \rightarrow G/H$ be the induced endomorphism. Denote by $\phi_1 : K \rightarrow K$ the endomorphism $\phi_1 = \iota \bar{\phi} \iota^{-1}$ of K , and let $\phi_2 = \phi \upharpoonright_H$. Then there exists a homomorphism $s_\phi : K \rightarrow H$ such that $\phi(x) = (\phi_1(x), s_\phi(x))$ for every $x \in K$. Now ϕ is the skew product of ϕ_1 and ϕ_2 via this s_ϕ .

A natural instance to this effect are fully invariant subgroups. For example, when D is a divisible group and $\phi \in \text{End}(D)$, then $H = t(D)$ is fully invariant, so necessarily ϕ -invariant. Thus, ϕ is a skew product.

In the sequel, for a skew product $\phi : G = K \times H \rightarrow K \times H$, we denote by $\phi_1 : K \rightarrow K$ the endomorphism of K conjugated to the induced endomorphism $\bar{\phi} : G/H \rightarrow G/H$ and we let $\phi_2 = \phi \upharpoonright_H$. The direct product $\pi_\phi = \phi_1 \times \phi_2$ is the *direct product associated to the skew product* ϕ . We can extend to $G = K \times H$ the homomorphism $s_\phi : K \rightarrow H$ associated to the skew product, defining it to be 0 on H . This allows us to consider $s_\phi \in \text{End}(G \times H)$ and speak of the composition $s_\phi^2 = 0$, as well as $\phi = \pi_\phi + s_\phi$ in the ring $\text{End}(G)$. In other words, the difference $s_\phi = \phi - \pi_\phi$ measures how much the skew product ϕ fails to coincide with its associated direct product π_ϕ .

Example 3.5. Let K be a torsion-free abelian group and let T be a torsion abelian group. Then every $\phi \in \text{End}(K \times T)$ is a skew product.

Proposition 3.6. Let G be an abelian group and $\phi \in \text{End}(G)$. Assume that $G = K \times T$, with T torsion and ϕ -invariant, and suppose that ϕ is a skew product such that $s_\phi(K)$ is finite. Then $\text{AT}_h(G, \phi, T)$ holds.

Proof. Let $\bar{\phi} : G/T \rightarrow G/T$ be the endomorphism induced by ϕ . By Proposition 2.8(d), $h(\pi_\phi) = h(\phi_1) + h(\phi_2)$. Moreover, by definition, $\phi_2 = \phi \upharpoonright_T$ and ϕ_1 is conjugated to $\bar{\phi}$, so that $h(\bar{\phi}) = h(\phi_1)$ by Proposition 2.8(a). Then it suffices to prove that

$$h(\phi) = h(\pi_\phi). \quad (3.3)$$

Let $F \in [G]^{<\omega}$. Then $F \subseteq F_1 \times F_2$, for some $F_1 \in [K]^{<\omega}$ and $F_2 \in [T]^{<\omega}$. We can assume without loss of generality that $(0, 0) \in F_1 \times F_2$ and that F_2 is a subgroup of T with $F_2 \supseteq s_\phi(K)$. To conclude the proof, it suffices to show that, for $n \in \mathbb{N}_+$,

$$T_n(\pi_\phi, F_1 \times F_2) = T_n(\phi, F_1 \times F_2). \quad (3.4)$$

We have $\pi_\phi^n(F_1 \times F_2) = \phi_1^n(F_1) \times \phi_2^n(F_2)$ and so $T_n(\pi_\phi, F_1 \times F_2) = T_n(\phi_1, F_1) \times T_n(\phi_2, F_2)$.

One can prove by induction that, for every $x \in K$ and every $n \in \mathbb{N}_+$,

$$\phi^n(x, 0) = (\phi_1^n(x), \phi_2^{n-1}(s_\phi(x)) + \phi_2^{n-2}(s_\phi(\phi_1(x))) + \dots + \phi_2(s_\phi(\phi_1^{n-2}(x))) + s_\phi(\phi_1^{n-1}(x))).$$

Since $s_\phi(K) \subseteq F_2$, we conclude that

$$\phi^n(x, 0) \in (\phi_1^n(x), 0) + [0 \times (\phi_2^{n-1}(F_2) + \phi_2^{n-2}(F_2) + \dots + \phi_2(F_2) + F_2)] = (\phi_1^n(x), 0) + [0 \times T_n(\phi_2, F_2)];$$

as $0 \times T_n(\phi_2, F_2) = T_n(\phi, 0 \times F_2)$ is a subgroup of G , we deduce

$$\phi^n(x, 0) \in (\phi_1^n(x), 0) + [0 \times T_n(\phi_2, F_2)] \text{ and } (\phi_1^n(x), 0) \in \phi^n(x, 0) + T_n(\phi, 0 \times F_2). \quad (3.5)$$

Fix $m \in \mathbb{N}$ and an m -tuple $a_0, a_1, \dots, a_{m-1} \in F_1$. Applying (3.5) to a_n for $n = 0, 1, \dots, m-1$ we get

$$\sum_{n=0}^{m-1} \phi^n(a_n, 0) \in \sum_{n=0}^{m-1} (\phi_1^n(a_n), 0) + [0 \times T_m(\phi_2, F_2)] \subseteq T_m(\phi_1, F_1) \times T_m(\phi_2, F_2)$$

and

$$\sum_{n=0}^{m-1} (\phi_1^n(a_n), 0) \in \sum_{n=0}^{m-1} \phi^n(a_n, 0) + T_m(\phi, 0 \times F_2) \subseteq T_m(\phi, F_1 \times 0) + T_m(\phi, 0 \times F_2).$$

In other words,

$$T_m(\phi, F_1 \times 0) \subseteq T_m(\phi_1, F_1) \times T_m(\phi_2, F_2) \text{ and } T_m(\phi_1, F_1) \times 0 \subseteq T_m(\phi, F_1 \times 0) + T_m(\phi, 0 \times F_2).$$

As $0 \times T_m(\phi_2, F_2) = T_m(\phi, 0 \times F_2)$ is a subgroup of G , we can write it also as

$$T_m(\phi, F_1 \times F_2) = T_m(\phi, F_1 \times 0) + T_m(\phi, 0 \times F_2) \subseteq T_m(\phi_1, F_1) \times T_m(\phi_2, F_2) = T_m(\pi_\phi, F_1 \times F_2)$$

and

$$T_m(\pi_\phi, F_1 \times F_2) = T_m(\phi_1, F_1) \times T_m(\phi_2, F_2) \subseteq T_m(\phi, F_1 \times 0) + T_m(\phi, 0 \times F_2) = T_m(\phi, F_1 \times F_2).$$

This proves (3.4), which gives the thesis. \square

Let G be an abelian group, $\phi \in \text{End}(G)$ and H, K ϕ -invariant subgroups of G . Let $\pi : G \rightarrow G/H$ be the canonical projection and $\bar{\phi} : G/H \rightarrow G/H$ the endomorphism induced on the quotient by ϕ . Then the subgroup $\pi(K)$ of G/H is $\bar{\phi}$ -invariant. Since $\pi(K)$ is isomorphic to $K/H \cap K$, and the induced endomorphism $\bar{\phi} \upharpoonright_K : K/(H \cap K) \rightarrow K/(H \cap K)$ is conjugated to $\bar{\phi} \upharpoonright_{\pi(K)}$ through this isomorphism,

$$h(\bar{\phi} \upharpoonright_{\pi(K)}) = h(\bar{\phi} \upharpoonright_K) \quad (3.6)$$

holds by Proposition 2.8(a).

In the sequel \wedge stays for conjunction.

Proposition 3.7. *Let G be an abelian group, $\phi \in \text{End}(G)$ and H, K ϕ -invariant subgroups of G . Then:*

$$\begin{aligned} & \text{AT}_h(G, \phi, K) \wedge \text{AT}_h(H, \phi \upharpoonright_H, H \cap K) \wedge \text{AT}_h(K, \phi \upharpoonright_K, H \cap K) \wedge \\ & \wedge \text{AT}_h(G/H, \bar{\phi}_H, (H+K)/H) \wedge \text{AT}_h(G/K, \bar{\phi}_K, (H+K)/K) \implies \text{AT}_h(G, \phi, H). \end{aligned}$$

Proof. The situation is described by the following diagram involving the pairs (G, H) and $(G/K, (H+K)/K)$:

$$\begin{array}{ccc} H & \xrightarrow{\phi \upharpoonright_H} & H \\ \downarrow & & \downarrow \\ G & \xrightarrow{\phi} & G \\ \downarrow & & \downarrow \\ G/H & \xrightarrow{\bar{\phi}_H} & G/H \end{array} \quad \begin{array}{ccc} (H+K)/K & \xrightarrow{\bar{\phi} \upharpoonright_{(H+K)/K}} & (H+K)/K \\ \downarrow & & \downarrow \\ G/K & \xrightarrow{\bar{\phi}_K} & G/K \\ \downarrow & & \downarrow \\ (G/K)/((H+K)/K) & \xrightarrow{\bar{\phi}_K} & (G/K)/((H+K)/K) \end{array}$$

Our hypotheses imply that:

- (i) $h(\phi) = h(\phi \upharpoonright_K) + h(\bar{\phi}_K)$;
- (ii) $h(\phi \upharpoonright_H) = h(\phi \upharpoonright_{H \cap K}) + h(\bar{\phi} \upharpoonright_H)$;

- (iii) $h(\phi \upharpoonright_K) = h(\phi \upharpoonright_{H \cap K}) + h(\overline{\phi \upharpoonright_K})$;
- (iv) $h(\overline{\phi_H}) = h(\overline{\phi_H} \upharpoonright_{(H+K)/H}) + h(\overline{\overline{\phi_H}})$;
- (v) $h(\overline{\phi_K}) = h(\overline{\phi_K} \upharpoonright_{(H+K)/K}) + h(\overline{\overline{\phi_K}})$.

The isomorphism $(G/K)/((H+K)/K) \cong (G/H)/((H+K)/H)$ commutes with

$$\overline{\overline{\phi_K}} : (G/K)/((H+K)/K) \rightarrow (G/K)/((H+K)/K) \text{ and } \overline{\overline{\phi_H}} : (G/H)/((H+K)/H) \rightarrow (G/H)/((H+K)/H),$$

so we get

$$h(\overline{\overline{\phi_H}}) = h(\overline{\overline{\phi_K}}), \quad (3.7)$$

applying Proposition 2.8(c). Moreover

$$h(\overline{\phi \upharpoonright_H}) = h(\overline{\phi} \upharpoonright_{H+K/K}) \text{ and } h(\overline{\phi \upharpoonright_K}) = h(\overline{\phi} \upharpoonright_{H+K/H}) \quad (3.8)$$

by (3.6). Applying (i), (iii), (v), (3.7), (3.8), (ii) and (iv),

$$\begin{aligned} h(\phi) &= h(\phi \upharpoonright_K) + h(\overline{\phi_K}) \\ &= (h(\phi \upharpoonright_{H \cap K}) + h(\overline{\phi \upharpoonright_K})) + (h(\overline{\phi_K} \upharpoonright_{(H+K)/K}) + h(\overline{\overline{\phi_K}})) \\ &= h(\phi \upharpoonright_{H \cap K}) + h(\overline{\phi} \upharpoonright_{H+K/H}) + h(\overline{\phi \upharpoonright_H}) + h(\overline{\overline{\phi_H}}) \\ &= (h(\phi \upharpoonright_{H \cap K}) + h(\overline{\phi \upharpoonright_H})) + (h(\overline{\phi} \upharpoonright_{H+K/H}) + h(\overline{\overline{\phi_H}})) \\ &= h(\phi \upharpoonright_H) + h(\overline{\phi_H}). \end{aligned}$$

Then $h(\phi) = h(\phi \upharpoonright_H) + h(\overline{\phi_H})$. □

Corollary 3.8. *Let G be an abelian group and $\phi \in \text{End}(G)$ and H, K ϕ -invariant subgroups of G with $H \subseteq K$. Then:*

- (a) $\text{AT}_h(G, \phi, H) \wedge \text{AT}_h(K, \phi \upharpoonright_K, H) \wedge \text{AT}_h(G/H, \overline{\phi}, K/H) \implies \text{AT}_h(G, \phi, K)$; and
- (b) $\text{AT}_h(G, \phi, K) \wedge \text{AT}_h(K, \phi \upharpoonright_K, H) \wedge \text{AT}_h(G/H, \overline{\phi}, K/H) \implies \text{AT}_h(G, \phi, H)$.

Corollary 3.9. *Let G be an abelian group, $\phi \in \text{End}(G)$ and H, K ϕ -invariant subgroups of G such that $G = H + K$. Then*

- (a) $\text{AT}_h(G, \phi, H \cap K) \wedge \text{AT}_h(H, \phi \upharpoonright_H, H \cap K) \implies \text{AT}_h(G, \phi, H)$;
- (b) $\text{AT}_h(G, \phi, K) \wedge \text{AT}_h(H, \phi \upharpoonright_H, H \cap K) \wedge \text{AT}_h(K, \phi \upharpoonright_K, H \cap K) \implies \text{AT}_h(G, \phi, H)$.

Proof. (a) We are going to apply Corollary 3.8(a) to the triple $H \cap K \subseteq H \subseteq G$. By hypothesis $\text{AT}_h(G, \phi, H \cap K)$ and $\text{AT}_h(H, \phi \upharpoonright_H, H \cap K)$ hold. For the last triple $(G/H \cap K, \overline{\phi}, H/H \cap K)$ note that $H/H \cap K$ is a ϕ -invariant subgroup of $G/H \cap K$ and $G/H \cap K \cong t(G)/H \cap K \times H/H \cap K$ is a splitting of $G/H \cap K$ into a direct sum of two ϕ -invariant subgroups. Therefore, $\text{AT}_h(G/H \cap K, \overline{\phi}, H/H \cap K)$ holds, as the $\overline{\phi}$ -invariant subgroup $H/H \cap K$ has a direct summand that is a $\overline{\phi}$ -invariant subgroup. □

(b) is obvious from Proposition 3.7. □

4 The Addition Theorem in the torsion-free case

The next properties, frequently used in the sequel, are easy to prove.

Lemma 4.1. *Let G be an abelian group and H a subgroup of G .*

- (a) *If H is divisible, then H is pure.*
- (b) *If G is divisible, then H is divisible if and only if H is pure.*
- (c) *If G is torsion-free, then the subgroup H is pure in G if and only if G/H is torsion-free.*

We start by showing that the Addition Theorem 1.3 holds for automorphisms of \mathbb{Q}^n and its invariant divisible subgroups, applying the Algebraic Yuzvinski Formula (1.4).

Proposition 4.2. *Let $m \in \mathbb{N}_+$, $\phi \in \text{Aut}(\mathbb{Q}^m)$ and H be a pure (i.e., divisible) ϕ -invariant subgroup of \mathbb{Q}^m . Then $\text{AT}_h(\mathbb{Q}^m, \phi, H)$ holds.*

Proof. Let $D = \mathbb{Q}^m$ and $r(H) = k \in \mathbb{N}$, that is, $H \cong \mathbb{Q}^k$. Then one can choose a basis $\mathcal{B} = \{v_1, \dots, v_k, v_{k+1}, \dots, v_m\}$ of D such that $\mathcal{B}_H = \{v_1, \dots, v_k\}$ is a basis of H and the matrix of ϕ with respect to \mathcal{B} has the following block-wise form:

$$A = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix},$$

where A_1 is the matrix of $\phi \upharpoonright_H$ with respect to \mathcal{B}_H . Let $\pi : D \rightarrow D/H$ be the canonical projection and $\bar{\phi} : D/H \rightarrow D/H$ the endomorphism induced by ϕ . Let $\bar{\mathcal{B}} = \{\pi(v_{k+1}), \dots, \pi(v_m)\}$, which is a basis of $D/H \cong \mathbb{Q}^{m-k}$. Then A_2 is the matrix of $\bar{\phi}$ with respect to $\bar{\mathcal{B}}$. Let $\alpha_1, \dots, \alpha_k$ be the eigenvalues of A_1 and let $\alpha_{k+1}, \dots, \alpha_m$ be the eigenvalues of A_2 . Then $\alpha_1, \dots, \alpha_n$ are the eigenvalues of A .

Let χ and $\chi_1, \chi_2 \in \mathbb{Q}[x]$ be the characteristic polynomials of A and A_1, A_2 respectively. Then $\chi = \chi_1 \chi_2$. Let s_1 and s_2 be the least common multiples of the denominators of the coefficients of χ_1 and χ_2 respectively. This means that $p_1 = s_1 \chi_1$ and $p_2 = s_2 \chi_2 \in \mathbb{Z}[x]$ are primitive. By Gauss Lemma $p = p_1 p_2$ is primitive and so for $s = s_1 s_2$ the polynomial $p = s \chi \in \mathbb{Z}[x]$ is primitive. Now the Algebraic Yuzvinski Formula (1.4) applied to $\phi, \phi \upharpoonright_H, \bar{\phi}$ gives

$$\begin{aligned} h(\phi) &= \log s + \sum_{1 \leq i \leq m, |\alpha_i| > 1} \log |\alpha_i| \\ &= \log(s_1 s_2) + \sum_{1 \leq i \leq k, |\alpha_i| > 1} \log |\alpha_i| + \sum_{k+1 \leq i \leq m, |\alpha_i| > 1} \log |\alpha_i| \\ &= \left(\log s_1 + \sum_{1 \leq i \leq k, |\alpha_i| > 1} \log |\alpha_i| \right) + \left(\log s_2 + \sum_{k+1 \leq i \leq m, |\alpha_i| > 1} \log |\alpha_i| \right) \\ &= h(\phi \upharpoonright_H) + h(\bar{\phi}). \end{aligned}$$

□

The next is a consequence of Corollary 2.14.

Corollary 4.3. *Let G be a torsion-free abelian group, $\phi \in \text{End}(G)$ and H a ϕ -invariant essential subgroup of G . Then $\text{AT}_h(G, \phi, H)$ holds.*

Proof. By Corollary 2.14(a), $h(\phi) = h(\phi \upharpoonright_H)$. If $h(\phi) = \infty$, this implies that $\text{AT}_h(G, \phi, H)$ holds. If $h(\phi) < \infty$, then $h(\bar{\phi}) = 0$ by Corollary 2.14(b) and so $\text{AT}_h(G, \phi, H)$ holds as well. □

An immediate consequence of this corollary is that for G a torsion-free abelian group, $\phi \in \text{End}(G)$, H a ϕ -invariant subgroup of G , and $\bar{\phi} : G/H \rightarrow G/H$ the endomorphism induced by ϕ , G/H is torsion implies that $\text{AT}_h(G, \phi, H)$ holds. Indeed, G/H torsion yields H essential in G .

The next is another consequence of Proposition 2.12 and Corollary 2.14. It shows that the verification of the Addition Theorem 1.3 for torsion-free abelian groups and their pure subgroups can be reduced to the case of divisible ones.

Corollary 4.4. *Let G be a torsion-free abelian group, $\phi \in \text{End}(G)$ and H a pure ϕ -invariant subgroup of G . Then $h(\phi) = h(\tilde{\phi})$, $h(\phi \upharpoonright_H) = h(\tilde{\phi} \upharpoonright_{D(H)})$ and $h(\bar{\phi}) = h(\bar{\tilde{\phi}})$, where $\bar{\phi} : G/H \rightarrow G/H$ and $\bar{\tilde{\phi}} : D(G)/D(H) \rightarrow D(G)/D(H)$ are the induced endomorphism. In particular, $\text{AT}_h(G, \phi, H)$ holds if and only if $\text{AT}_h(D(G), \tilde{\phi}, D(H))$ holds.*

Proof. Since the purification H_* of H in $D(G)$ is divisible in view of Lemma 4.1, $H_* = D(H)$ and $H = D(H) \cap G$. By Proposition 2.12, $h(\phi) = h(\tilde{\phi})$ and $h(\phi \upharpoonright_H) = h(\tilde{\phi} \upharpoonright_{D(H)})$. Let $\pi : D(G) \rightarrow D(G)/D(H)$ be the canonical projection. Then $\pi(G)$ is essential in $D(G)/D(H)$ and $h(\bar{\phi}) = h(\bar{\tilde{\phi}} \upharpoonright_{\pi(G)})$ by Corollary 2.14(a). Since $G/H \cong \pi(G)$, and $\bar{\phi}$ is conjugated to $\bar{\tilde{\phi}} \upharpoonright_{\pi(G)}$ through this isomorphism, $h(\bar{\phi}) = h(\bar{\tilde{\phi}} \upharpoonright_{\pi(G)})$ by Proposition 2.8(a). Hence $h(\bar{\phi}) = h(\bar{\tilde{\phi}})$. □

If the abelian group G is torsion-free, then for $\phi \in \text{End}(G)$ the subgroup $\ker_\infty \phi$ is also pure. The next result will reduce the computation of the entropy of endomorphisms of a finite-rank torsion-free divisible abelian groups to the case of *injective* ones.

Proposition 4.5. *Let G be a torsion-free abelian group of finite rank and $\phi \in \text{End}(G)$. Then $h(\phi) = h(\bar{\phi})$, where $\bar{\phi} : G/\ker_\infty \phi \rightarrow G/\ker_\infty \phi$. Moreover, $\text{AT}_h(G, \phi, \ker_\infty \phi)$ holds.*

Proof. Suppose first that $G = D$ is divisible. Since D has finite rank, $\ker_\infty \phi$ has finite rank as well. Then there exists $n \in \mathbb{N}$, such that $\ker_\infty \phi = \ker \phi^n$. Let $\gamma = \phi^n$ and $\bar{\gamma} : D/\ker \gamma \rightarrow D/\ker \gamma$ the endomorphism induced by γ . Then $h(\gamma) = nh(\phi)$ by Proposition 2.8(b). Since $\bar{\gamma} = \bar{\phi}^n$, it follows that $h(\bar{\gamma}) = nh(\bar{\phi})$ again by Proposition 2.8(b). So, if we prove that $h(\gamma) = h(\bar{\gamma})$, it will follow that $h(\phi) = h(\bar{\phi})$.

This shows that we can suppose without loss of generality that $\ker_\infty \phi = \ker \phi$; let $\bar{\phi} : D/\ker \phi \rightarrow D/\ker \phi$ be the endomorphism induced by ϕ . From [10, Section 58, Theorem 1] it follows that

$$D \cong \ker \phi \times \phi(D). \quad (4.1)$$

Corollary 2.6 gives $h(\phi \upharpoonright_{\ker \phi}) = 0$ and so $h(\phi) = h(\phi \upharpoonright_{\phi(D)})$ by Proposition 2.8(d). Since $\phi(D) \cong D/\ker \phi$, and $\phi \upharpoonright_{\phi(D)}$ and $\bar{\phi}$ are conjugated by this isomorphism, $h(\phi \upharpoonright_{\phi(D)}) = h(\bar{\phi})$ by Proposition 2.8(a). Hence $h(\phi) = h(\bar{\phi})$.

We consider now the general case. Since $\ker_\infty \tilde{\phi}$ is pure in $D(G)$, it is divisible by Lemma 4.1. Moreover $\ker_\infty \phi$ is essential in $\ker_\infty \tilde{\phi}$. Indeed, let $x \in \ker_\infty \tilde{\phi}$, i.e., there exists $n \in \mathbb{N}_+$ such that $\tilde{\phi}^n(x) = 0$. Since G is essential in $D(G)$ there exists $k \in \mathbb{Z}$ such that $kx \in G \setminus \{0\}$. Moreover, $\phi^n(kx) = \tilde{\phi}^n(kx) = k\tilde{\phi}^n(x) = 0$ and so $kx \in \ker_\infty \phi \setminus \{0\}$. It follows that $\ker_\infty \tilde{\phi} = D(\ker_\infty \phi)$. By the first part of the proof, that is, by the divisible case, $h(\tilde{\phi}) = h(\bar{\tilde{\phi}})$, where $\bar{\tilde{\phi}} : D(G)/\ker_\infty \tilde{\phi} \rightarrow D(G)/\ker_\infty \tilde{\phi}$ is the induced endomorphism. By Proposition 2.12 $h(\phi) = h(\tilde{\phi})$ and by Corollary 4.4 $h(\bar{\phi}) = h(\bar{\tilde{\phi}})$. Hence $h(\phi) = h(\bar{\phi})$. \square

We can prove now that the Addition Theorem 1.3 holds in the case of torsion-free countable finite-rank abelian groups and *pure* invariant subgroups. Indeed, the next proposition generalizes Proposition 4.2 to endomorphisms. As far as we work in the torsion-free context, it seems natural to consider mainly pure subgroups that allows us to remain in the class even under passage to quotients.

Proposition 4.6. *Let $n \in \mathbb{N}_+$, $\phi \in \text{End}(\mathbb{Q}^n)$ and H a pure (i.e., divisible) ϕ -invariant subgroup of \mathbb{Q}^n . Then $h(\phi) < \infty$ and $\text{AT}_h(\mathbb{Q}^n, \phi, H)$ holds.*

Proof. Let $D = \mathbb{Q}^n$. We prove first that we can assume without loss of generality that ϕ is injective. Indeed, consider $\bar{\phi} : D/\ker_\infty \phi \rightarrow D/\ker_\infty \phi$. Then $\bar{\phi}$ is injective and $D/\ker_\infty \phi$ is divisible and torsion-free as $\ker_\infty \phi$ is pure in D (see Lemma 4.1(c)). Therefore, $D/\ker_\infty \phi \cong \mathbb{Q}^m$ for some $m \in \mathbb{N}$, $m \leq n$, and $\bar{\phi}$ is an automorphism of $D/\ker_\infty \phi$. By Proposition 4.5 $h(\phi) = h(\bar{\phi})$. Hence $h(\phi) < \infty$ by Theorem 1.4.

Let $\pi : D \rightarrow D/\ker_\infty \phi$ be the canonical projection. Then $\pi(H) = (H + \ker_\infty \phi)/\ker_\infty \phi$ is a $\bar{\phi}$ -invariant pure (i.e., divisible) subgroup of $D/\ker_\infty \phi$ and we have the following two diagrams, where $\bar{\phi}_H : D/H \rightarrow D/H$ is the endomorphism induced by ϕ and $\bar{\phi}_{\pi(H)} : (D/\ker_\infty \phi)/\pi(H) \rightarrow (D/\ker_\infty \phi)/\pi(H)$ is the endomorphism induced by $\bar{\phi}$.

$$\begin{array}{ccc} H & \xrightarrow{\phi \upharpoonright_H} & H \\ \downarrow & & \downarrow \\ D & \xrightarrow{\phi} & D \\ \downarrow & & \downarrow \\ D/H & \xrightarrow{\bar{\phi}_H} & D/H \end{array} \quad \begin{array}{ccc} \pi(H) & \xrightarrow{\bar{\phi} \upharpoonright_{\pi(H)}} & \pi(H) \\ \downarrow & & \downarrow \\ D/\ker_\infty \phi & \xrightarrow{\bar{\phi}} & D/\ker_\infty \phi \\ \downarrow & & \downarrow \\ (D/\ker_\infty \phi)/\pi(H) & \xrightarrow{\bar{\phi}_{\pi(H)}} & (D/\ker_\infty \phi)/\pi(H) \end{array}$$

By Proposition 4.5

(i) $\text{AT}_h(D, \phi, \ker_\infty \phi)$ holds.

Since $\ker_\infty \phi \upharpoonright_H = H \cap \ker_\infty \phi$ is $\phi \upharpoonright_H$ -invariant and is pure in H , by Proposition 4.5

(ii) $\text{AT}_h(H, \phi \upharpoonright_H, H \cap \ker_\infty \phi)$ holds.

Since $h(\phi \upharpoonright_{\ker_\infty \phi}) = 0$ by Corollary 2.6,

(iii) $\text{AT}_h(\ker_\infty \phi, \phi \upharpoonright_{\ker_\infty \phi}, H \cap \ker_\infty \phi)$ holds.

Since $\ker_\infty \bar{\phi} = (H + \ker_\infty \phi)/H$, by Proposition 4.5,

(iv) $\text{AT}_h(D/H, \bar{\phi}_H, (H + \ker_\infty \phi)/H)$ holds.

Assume that

(v) $\text{AT}_h(D/\ker_\infty \phi, \overline{\phi}, (H + \ker_\infty \phi)/\ker_\infty \phi)$ holds.

So the hypotheses of Proposition 3.7 are satisfied and it yields that $\text{AT}_h(D, \phi, H)$ holds. This shows that we can assume without loss of generality that ϕ is injective. Then ϕ is surjective as well, hence the conclusion follows from Proposition 4.2. \square

The following is a clear consequence of Proposition 4.6 and Corollary 4.4.

Corollary 4.7. *Let G be a torsion-free abelian group of finite rank, $\phi \in \text{End}(G)$ and H a pure ϕ -invariant subgroup of G . Then $\text{AT}_h(G, \phi, H)$ holds.*

In the remaining part of this section we discuss other consequences of Propositions 2.12 and 4.6.

Corollary 4.8. *Let G be a torsion-free abelian group and $\phi \in \text{End}(G)$.*

(a) *If G has finite rank, then $h(\phi) < \infty$.*

(b) *If $G = V(\phi, g)$ for some $g \in G$, then G has finite rank if and only if $h(\phi) < \infty$.*

Proof. (a) Since $D(G) \cong \mathbb{Q}^n$, for $n \in \mathbb{N}_+$, Proposition 4.6 gives $h(\tilde{\phi}) < \infty$, and so $h(\phi) = h(\tilde{\phi}) < \infty$ by Proposition 2.12.

(b) When $G = V(\phi, g)$ has finite rank, then $h(\phi) < \infty$ by (a). On the other hand, $h(\phi) < \infty$ implies $r(G)$ finite by Lemma 2.11. \square

Corollary 4.8(b) allows for a clear picture about the entropies of endomorphisms of torsion-free abelian groups. Indeed, for every element g of a torsion-free abelian group G ,

$$h(\phi \upharpoonright_{V(\phi, g)}) = \infty \text{ if and only if } r(V(\phi, g)) \text{ is infinite.}$$

Corollary 4.9. *Let G be a torsion-free abelian group and $\phi \in \text{End}(G)$. If $h(\phi \upharpoonright_{V(\phi, g)}) = \infty$, then $h(\phi \upharpoonright_{V(\phi, z)}) = \infty$ for every $z \in V(\phi, g) \setminus \{0\}$.*

Proof. By Corollary 4.8(b), $h(\phi \upharpoonright_{V(\phi, g)}) = \infty$ implies $r(V(\phi, g))$ infinite. For $z \in V(\phi, g) \setminus \{0\}$, it is easy to see that $r(V(\phi, z))$ is infinite as well, and so $h(\phi \upharpoonright_{V(\phi, z)}) = \infty$ again by Corollary 4.8(b). \square

In other words, for every ϕ -invariant subgroup H of $V(\phi, g)$ one has the following surprising dichotomy:

$$\text{either } H = 0 \text{ or } h(\phi \upharpoonright_H) = \infty.$$

5 Proof of the Addition Theorem

Lemma 5.1. *Let G be a torsion-free abelian group of finite rank. Then mG has finite index in G for every $m \in \mathbb{N}_+$.*

Proof. Let $n = r(G)$. Let $m \in \mathbb{N}_+$ and write it as product of primes, that is, $m = p_1 \cdot \dots \cdot p_r$. We proceed by induction on $r \in \mathbb{N}_+$. Let $r = 1$, that is, $m = p$ is a prime. Since $r(pG) = n$, we can think that $\mathbb{Z}^n \subseteq pG \subseteq \mathbb{Q}^n$. Consequently, $G/pG \cong (G/\mathbb{Z}^n)/(pG/\mathbb{Z}^n)$. Since $G/\mathbb{Z}^n \subseteq \mathbb{Q}^n/\mathbb{Z}^n \cong \bigoplus_q (\mathbb{Z}(q^\infty))^n$, $G/\mathbb{Z}^n \cong \bigoplus_q F_q$, where $F_q \leq \mathbb{Z}(q^\infty)^n$ for every prime q . Then $F_q = \mathbb{Z}(q^\infty)^{n_q} \times L_q$, for some $n_q \in \mathbb{N}$ with $n_q \leq n$ and $L_q \leq F_q$ finite (see [7, Section 25.1]). For every prime $q \neq p$, $pF_q = F_q$, so $G/pG \cong (\bigoplus_q F_q)/(pF_p \times \bigoplus_{q \neq p} F_q) \cong F_p/pF_p \cong L_p/pL_p$. Now L_p is finite and so $G/pG \cong L_p/pL_p$ is finite as well, and this shows that pG has finite index in G .

Assume now that the assertion holds for r and that $m = p_1 \cdot \dots \cdot p_{r+1}$. By inductive hypothesis, $G' = p_2 \cdot \dots \cdot p_{r+1}G$ has finite index in G , and by the case $r = 1$, $p_1 G'$ has finite index in G' . Then mG has finite index in G . \square

It is now possible to prove in the following proposition that the Addition Theorem 1.3 holds with respect to the torsion subgroup. We assume that the group is countable, but this hypothesis is removable in view of Proposition 3.3.

Proposition 5.2. *Let G be a countable abelian group and $\phi \in \text{End}(G)$. Then $\text{AT}_h(G, \phi, t(G))$ holds.*

Proof. Suppose that there exists $g \in G$ such that $V(\phi, g)$ has infinite rank. Let $\pi : G \rightarrow G/t(G)$ be the canonical projection on the torsion-free abelian group $G/t(G)$, and let $\bar{\phi} : G/t(G) \rightarrow G/t(G)$ be the induced endomorphism. Then $\pi(V(\phi, g)) = V(\bar{\phi}, \pi(g))$ has infinite rank and so $h(\bar{\phi} \upharpoonright_{V(\bar{\phi}, \pi(g))}) = \infty$ by Lemma 2.11. Therefore, $h(\bar{\phi}) = \infty$ and $h(\phi) = \infty$ by Lemma 2.7.

Suppose now that $V(\phi, g)$ has finite rank for every $g \in G$. Let $F \in [G]^{<\omega}$. Then

$$G_F = V(\phi, F)$$

has finite rank and we show that

$$\text{AT}_h(G_F, \phi \upharpoonright_{G_F}, t(G_F)) \text{ holds.} \quad (5.1)$$

By Lemma 2.17(c) there exists $m \in \mathbb{N}_+$ such that $mt(G_F) = 0$. Then there exists a finite-rank torsion-free subgroup K of G_F such that $G_F \cong K \times t(G_F)$ (in view of a theorem by Kulikov, see [7, Section 27.5]). Since $t(G_F)$ is a ϕ -invariant subgroup of G_F that splits, by Remark 3.4 this gives rise to a skew product, that is, there exists a homomorphism $s_\phi : K \rightarrow t(G_F)$ such that $\phi(x, y) = (\phi_1(x), \phi_2(y) + s_\phi(x))$ for every $(x, y) \in G_F$, where $\phi_1 : K \rightarrow K$ is conjugated to $\bar{\phi} : G_F/t(G_F) \rightarrow G_F/t(G_F)$ by the isomorphism $K \cong G_F/t(G_F)$, and $\phi_2 = \phi \upharpoonright_{t(G_F)}$. We show that $s_\phi(K)$ is finite. In fact, let $\pi : K \rightarrow K/mK$ be the canonical projection and let $\psi : K/mK \rightarrow t(G_F)$ be defined by $\psi(\pi(x)) = s_\phi(x)$ for every $x \in K$. Since $s_\phi(mK) = ms_\phi(K) \subseteq mt(G_F) = 0$, so $s_\phi(mK) = 0$. Thus ψ is well-defined and $s_\phi = \psi \circ \pi$. Now K/mK is finite by Lemma 5.1, so $s_\phi(G_F) = \psi(K/mK)$ is finite as well. Then Proposition 3.6 gives (5.1), which is equivalent to

$$h(\phi \upharpoonright_{G_F}) = h(\phi \upharpoonright_{t(G_F)}) + h(\overline{\phi \upharpoonright_{G_F}}), \quad (5.2)$$

where $\overline{\phi \upharpoonright_{G_F}} : G_F/t(G_F) \rightarrow G_F/t(G_F)$ is the induced endomorphism. By (3.6)

$$h(\overline{\phi \upharpoonright_{G_F}}) = h(\bar{\phi} \upharpoonright_{G_F/t(G_F)}).$$

By Proposition 2.19 there exists $\{L_n : n \in \mathbb{N}\} \subseteq [G]^{<\omega}$, such that:

$$\begin{aligned} h(\phi) &= \lim_{n \rightarrow \infty} h(\phi \upharpoonright_{V(\phi, L_n)}), \\ h(\phi \upharpoonright_{t(G)}) &= \lim_{n \rightarrow \infty} h(\phi \upharpoonright_{t(G) \cap V(\phi, L_n)}) = \lim_{n \rightarrow \infty} h(\phi \upharpoonright_{t(V(\phi, L_n))}), \text{ and} \\ h(\bar{\phi}) &= \lim_{n \rightarrow \infty} h(\bar{\phi} \upharpoonright_{\pi(L_n)}) = \lim_{n \rightarrow \infty} h(\overline{\phi \upharpoonright_{V(\phi, L_n)}}). \end{aligned}$$

Consequently, by (5.2) and these equalities,

$$\begin{aligned} h(\phi) &= \lim_{n \rightarrow \infty} h(\phi \upharpoonright_{V(\phi, L_n)}) \\ &= \lim_{n \rightarrow \infty} h(\phi \upharpoonright_{t(V(\phi, L_n))}) + \lim_{n \rightarrow \infty} h(\overline{\phi \upharpoonright_{V(\phi, L_n)}}) \\ &= h(\phi \upharpoonright_{t(G)}) + h(\bar{\phi}), \end{aligned}$$

that is, $\text{AT}_h(G, \phi, t(G))$ holds. \square

Lemma 5.3. *Let G be a countable abelian group, $\phi \in \text{End}(G)$ and H a ϕ -invariant subgroup of G such that G/H is torsion. Then $\text{AT}_h(G, \phi, H)$ holds.*

Proof. We will apply Proposition 3.7 with $K = t(G)$. Let $\bar{\phi}_H : G/H \rightarrow G/H$ and $\bar{\phi}_{t(G)} : G/t(G) \rightarrow G/t(G)$ be the endomorphisms induced by ϕ . We have that

- (i) $\text{AT}_h(G, \phi, t(G))$ and
- (ii) $\text{AT}_h(H, \phi \upharpoonright_H, t(H))$ hold by Proposition 5.2;
- (iii) $\text{AT}_h(t(G), \phi \upharpoonright_{t(G)}, t(H))$ and
- (iv) $\text{AT}_h(G/H, \bar{\phi}_H, (H + t(G))/H)$ hold because $t(G)$ and G/H are torsion [5];
- (v) $\text{AT}_h(G/t(G), \bar{\phi}_{t(G)}, (H + t(G))/t(G))$ holds by Corollary 4.3, as $G/t(G)$ is torsion-free and $(H + t(G))/t(G)$ is essential in $G/t(G)$, being $G/(H + t(G))$ torsion as a quotient of the torsion group G/H .

Now Proposition 3.7 with $K = t(G)$ applies to conclude the proof. \square

Proposition 5.4. *Let G be a countable torsion-free abelian group, $\phi \in \text{End}(G)$ and let H be a ϕ -invariant subgroup of G . Then $\text{AT}_h(G, \phi, H)$ holds true.*

Proof. Assume there exists $g \in G$ such that $V(\phi, g)$ has infinite rank. Then $h(\phi \upharpoonright_{V(\phi, g)}) = \infty$ by Corollary 4.8(b) and so $h(\phi) = \infty$ by Lemma 2.7. If $V(\phi, g) \cap H$ is non-zero, then $h(\phi \upharpoonright_H) = \infty$ by Corollary 4.9 and Lemma 2.7. Then $h(\phi) = h(\phi \upharpoonright_H) = \infty$ and in particular $\text{AT}_h(G, \phi, H)$ holds. Assume now that $V(\phi, g) \cap H = 0$. Let $\pi : G \rightarrow G/H$ be the canonical projection. Therefore, $V(\phi, g)$ projects injectively in the quotient G/H and so $\pi(V(\phi, g)) = V(\bar{\phi}, \pi(G))$ has infinite rank. So $h(\bar{\phi}) = \infty$ by Corollary 4.8(b) and Lemma 2.7. Hence $h(\phi) = h(\bar{\phi}) = \infty$ by Lemma 2.7 and in particular $\text{AT}_h(G, \phi, H)$ holds.

We show now that

$$\text{if } G \text{ has finite rank, then } \text{AT}_h(G, \phi, H) \text{ holds true.} \quad (5.3)$$

According to Proposition 2.12 $h(\bar{\phi}) = h(\phi)$. Since G is essential in $D(G)$, G/H is essential in $D(G)/H$. Let $\bar{\phi} : G/H \rightarrow G/H$ and $\bar{\phi} : D(G)/H \rightarrow D(G)/H$ be the induced endomorphisms, and note that $\bar{\phi} \upharpoonright_{G/H} = \bar{\phi}$. By Corollary 2.14(a), $h(\bar{\phi}) = h(\bar{\phi})$. Then it suffices to prove that $(D(G), \bar{\phi}, H)$ holds. Since $D(H)$ is ϕ -invariant and $D(H)/H = t(D(G)/H)$, hence $\text{AT}_h(D(G)/H, \bar{\phi}, D(G)/H)$ holds by Proposition 5.2. Moreover, $\text{AT}_h(D(H), \bar{\phi} \upharpoonright_{D(H)}, H)$ holds by Lemma 5.3, and $\text{AT}_h(D(G), \bar{\phi}, D(H))$ holds by Proposition 4.6. Therefore, Corollary 3.8(b) applies to conclude that $\text{AT}_h(D(G), \bar{\phi}, H)$ holds.

So, going back to the general case, we can suppose now that $V(\phi, g)$ has finite rank for every $g \in G$. In particular $V(\phi, F)$ has finite rank for every $F \in [G]^{<\omega}$. By (5.3) $\text{AT}_h(V(\phi, F), \phi \upharpoonright_{V(\phi, F)}, H \cap V(\phi, F))$ holds for every $F \in [G]^{<\omega}$, that is

$$h(\phi \upharpoonright_{V(\phi, F)}) = h(\phi \upharpoonright_{H \cap V(\phi, F)}) + h(\overline{\phi \upharpoonright_{V(\phi, F)}}),$$

where $\overline{\phi \upharpoonright_{V(\phi, F)}} : V(\phi, F)/(H \cap V(\phi, F)) \rightarrow V(\phi, F)/(H \cap V(\phi, F))$ is the induced homomorphism. By (3.6)

$$h(\overline{\phi \upharpoonright_{V(\phi, F)}}) = h(\bar{\phi} \upharpoonright_{V(\phi, F)/(H \cap V(\phi, F))}).$$

By Proposition 2.19 there exists a family $\{L_n : n \in \mathbb{N}\}$ of finite subsets of G , such that:

$$\begin{aligned} h(\phi) &= \lim_{n \rightarrow \infty} h(\phi \upharpoonright_{V(\phi, L_n)}), \\ h(\phi \upharpoonright_H) &= \lim_{n \rightarrow \infty} h(\phi \upharpoonright_{H \cap V(\phi, L_n)}), \text{ and} \\ h(\bar{\phi}) &= \lim_{n \rightarrow \infty} h(\bar{\phi} \upharpoonright_{\pi(L_n)}) = \lim_{n \rightarrow \infty} h(\overline{\phi \upharpoonright_{V(\phi, L_n)}}). \end{aligned}$$

Consequently

$$\begin{aligned} h(\phi) &= \lim_{n \rightarrow \infty} h(\phi \upharpoonright_{V(\phi, L_n)}) \\ &= \lim_{n \rightarrow \infty} h(\phi \upharpoonright_{H \cap V(\phi, L_n)}) + \lim_{n \rightarrow \infty} h(\overline{\phi \upharpoonright_{V(\phi, L_n)}}) \\ &= h(\phi \upharpoonright_H) + h(\bar{\phi}), \end{aligned}$$

and this gives the thesis. \square

Corollary 5.5. *Let G be a countable abelian group, $\phi \in \text{End}(G)$ and let H be a ϕ -invariant subgroup of G . Then $\text{AT}_h(G, \phi, H)$ holds when:*

- (a) H is a torsion subgroup of G ;
- (b) H contains the torsion subgroup $t(G)$;
- (c) there exists a ϕ -invariant subgroup K of G such that $H \cap K \subseteq t(G)$ and $G = H + K$;
- (d) $G = H + t(G)$.

Proof. (a) The subgroup $t(G)/H$ of G/H is precisely $t(G/H)$, so both $\text{AT}_h(G/H, \bar{\phi}, t(G)/H)$ and $\text{AT}_h(G, \phi, t(G))$ hold by our hypothesis. On the other hand, $\text{AT}_h(t(G), \phi \upharpoonright_{t(G)}, H)$ holds as $t(G)$ is torsion [5]. Now Corollary 3.8(b) applies to the triple $H \subseteq t(G) \subseteq G$.

(b) Note that $\text{AT}_h(G, \phi, t(G))$ and $\text{AT}_h(H, \phi \upharpoonright_H, t(G))$ hold by Proposition 5.2, and $\text{AT}_h(G/t(G), \bar{\phi}, H/t(H))$ holds by Proposition 5.4. By Corollary 3.8(a) applied to the triple $t(G) \subseteq H \subseteq G$, $\text{AT}_h(G, \phi, H)$ holds.

(c) Follows directly from (a) and Corollary 3.9(a).

(d) Follows immediately from (c) with $K = t(G)$. \square

We can now prove the Addition Theorem 1.3:

Proof of Theorem 1.3. By Proposition 3.3 we can suppose that G is countable. According to Corollary 5.5(b) we have that $\text{AT}_h(G, \phi, t(G) + H)$ holds, while $\text{AT}_h(t(G) + H, \phi \upharpoonright_{t(G)+H}, H)$ holds by Corollary 5.5(d). Finally, $\text{AT}_h(G/H, \bar{\phi}, (t(G) + H)/H)$ holds by Corollary 5.5(a) as the subgroup $(t(G) + H)/H$ of G/H is torsion. Now Corollary 3.8 applies to the triple $H \subseteq t(G) + H \subseteq G$ to conclude the proof. \square

6 The Uniqueness Theorem

We start this section proving the Uniqueness Theorem 1.5 for the algebraic entropy h in the category of all abelian groups.

Proof of Theorem 1.5. Let $h^* = \{h_G^* : G \text{ abelian group}\}$ be a collection of functions $h_G^* : \text{End}(G) \rightarrow \mathbb{R}_+$ satisfying (a) – (e) from Theorem 1.5. We have to show that $h^*(\phi) = h(\phi)$ for every abelian group G and $\phi \in \text{End}(G)$.

(i) If G is torsion, then $h^*(\phi) = \text{ent}(\phi) = h(\phi)$ by the Uniqueness Theorem for ent proved in [5, Theorem 6.1].

(ii) It suffices to consider the case when G is torsion-free. Indeed, let $\bar{\phi} : G/t(G) \rightarrow G/t(G)$ be the endomorphism induced by ϕ , where $G/t(G)$ is torsion-free. By (c) $h^*(\phi) = h^*(\phi \upharpoonright_{t(G)}) + h^*(\bar{\phi})$ and by the Addition Theorem 1.3 $h(\phi) = h(\phi \upharpoonright_{t(G)}) + h(\bar{\phi})$. Since $h^*(\phi \upharpoonright_{t(G)}) = h(\phi \upharpoonright_{t(G)})$ by (i), it follows that $h^*(\phi) = h(\phi)$ if $h^*(\bar{\phi}) = h(\bar{\phi})$.

(iii) If G is torsion-free and $r(G) = n$ is finite, then $h^*(\phi) = h(\phi)$. Indeed, $D(G) \cong \mathbb{Q}^n$ and $D(G)/G$ is torsion. Let $\bar{\phi} : D(G)/G \rightarrow D(G)/G$ be the endomorphism induced by ϕ . By the Algebraic Yuzvinski Formula (1.4) and by (e) $h^*(\bar{\phi}) = h(\bar{\phi})$ and the value is finite. Since $D(G)/G$ is torsion, $h^*(\bar{\phi}) = h(\bar{\phi})$ by (i). By (c) $h^*(\phi) = h^*(\phi) + h^*(\bar{\phi})$ and by the Addition Theorem 1.3 $h(\phi) = h(\phi) + h(\bar{\phi})$. Then $h^*(\phi) = h^*(\phi) + h^*(\bar{\phi}) = h(\phi) + h(\bar{\phi}) = h(\phi)$.

(iv) If G is torsion-free, $G = V(\phi, g)$ for some $g \in G$ and $r(G)$ is infinite, then $h^*(\phi) = \infty = h(\phi)$. In fact, $G \cong \bigoplus_{n \in \mathbb{N}} \langle \phi^n(g) \rangle \cong \bigoplus_{n \in \mathbb{N}} \mathbb{Z}$. Moreover, ϕ is conjugated to $\beta_{\mathbb{Z}}$ through this isomorphism and so $h(\phi) = \infty$ by Proposition 2.8(a) and Example 2.10. Also $h^*(\phi) = \infty$. Indeed, $h^*(\phi) = h^*(\beta_{\mathbb{Z}})$ by (a). Moreover, $h^*(\beta_{\mathbb{Z}(p)}) = \log p$, for every prime p , by (d). For $G = \mathbb{Z}^{(\mathbb{N})}$ and for every prime p , the subgroup pG of G is $\beta_{\mathbb{Z}}$ -invariant, so induces an endomorphism $\bar{\beta}_{\mathbb{Z}} : G/pG \rightarrow G/pG$. Since $G/pG \cong \mathbb{Z}(p)^{(\mathbb{N})}$, and $\bar{\beta}_{\mathbb{Z}}$ is conjugated to $\beta_{\mathbb{Z}(p)}$ through this isomorphism, $h^*(\bar{\beta}_{\mathbb{Z}}) = h^*(\beta_{\mathbb{Z}(p)}) = \log p$ by (a). Therefore, $h^*(\beta_{\mathbb{Z}}) \geq \log p$ for every prime p by (c), and so $h^*(\beta_{\mathbb{Z}}) = \infty$.

(v) If G is torsion-free and $G = V(\phi, F)$ for some $F \in [G]^{<\omega}$, then $h^*(\phi) = h(\phi)$. To prove this, let $F = \{f_1, \dots, f_k\}$. Then $G = V(\phi, f_1) + \dots + V(\phi, f_k)$. If $r(V(\phi, f_i))$ is finite for every $i \in \{1, \dots, k\}$, then $r(G)$ is finite as well, and $h^*(\phi) = h(\phi)$ by (iii). If $r(V(\phi, f_i))$ is infinite for some $i \in \{1, \dots, k\}$, then $h^*(\phi \upharpoonright_{V(\phi, f_i)}) = \infty = h(\phi \upharpoonright_{V(\phi, f_i)})$ by (iv). By (c) and by Lemma 2.7, we have $h^*(\phi) = \infty = h(\phi)$.

(vi) Consider now the general case. By (ii) we can suppose without loss of generality that G is torsion-free. By Lemma 2.18 $G = \varinjlim \{V(\phi, F) : F \in [G]^{<\omega}\}$. By (v) $h^*(\phi \upharpoonright_{V(\phi, F)}) = h(\phi \upharpoonright_{V(\phi, F)})$ for every $F \in [G]^{<\omega}$. Therefore, (b) and Proposition 2.8(c) give $h^*(\phi) = \sup_{F \in [G]^{<\omega}} h^*(\phi \upharpoonright_{V(\phi, F)}) = \sup_{F \in [G]^{<\omega}} h(\phi \upharpoonright_{V(\phi, F)}) = h(\phi)$. \square

Note that the logarithmic law is not among the properties necessary to give uniqueness of h in the category of all abelian groups. This is different from the behavior of ent. Moreover, this means that the logarithmic law follows automatically from the other properties.

It is possible to prove the Uniqueness Theorem 1.5 also in a less direct way, that is, using a known theorem by Vámos on length functions [27]. We explain this alternative proof in the remaining part of this section.

Let R be a unitary commutative ring. We denote by \mathbf{Mod}_R the category of all R -modules and their homomorphisms. An *invariant* $i : \mathbf{Mod}_R \rightarrow \mathbb{R}_+ \cup \{\infty\}$ is a function such that $i(0) = 0$ and $i(M) = i(M')$ if $M \cong M'$ in \mathbf{Mod}_R . For $M \in \mathbf{Mod}_R$, denote by $\mathcal{F}(M)$ the family of all finitely generated submodules of M .

Definition 6.1. [19, 27] Let R be a unitary commutative ring. A length function L of \mathbf{Mod}_R is an invariant $L : \mathbf{Mod}_R \rightarrow \mathbb{R}_+ \cup \{\infty\}$ such that:

(a) $L(M) = L(M') + L(M'')$ for every exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ in \mathbf{Mod}_R ;

(b) $L(M) = \sup\{L(F) : F \in \mathcal{F}(M)\}$.

An invariant satisfying (b) is said *additive* and an invariant with (c) is called *upper continuous*. So a length function is an additive upper continuous invariant of \mathbf{Mod}_R .

Consider the category \mathbf{AbGrp} of all abelian groups and their homomorphisms. As done more generally in [3], we introduce the category $\mathbf{Flow}_{\mathbf{AbGrp}}$ of flows of \mathbf{AbGrp} . The objects of $\mathbf{Flow}_{\mathbf{AbGrp}}$ (namely, the *algebraic flows*) are the pairs (G, ϕ) with $G \in \mathbf{AbGrp}$ and $\phi \in \text{End}(G)$. A morphism $u : (G, \phi) \rightarrow (H, \psi)$ in $\mathbf{Flow}_{\mathbf{AbGrp}}$ between two algebraic flows (G, ϕ) and (H, ψ) is an homomorphism $u : G \rightarrow H$ such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G \\ u \downarrow & & \downarrow u \\ H & \xrightarrow{\psi} & H \end{array} \quad (6.1)$$

in \mathbf{AbGrp} commutes. Two algebraic flows (G, ϕ) and (H, ψ) are isomorphic in $\mathbf{Flow}_{\mathbf{AbGrp}}$ if the morphism $u : G \rightarrow H$ in (6.1) is an isomorphism in \mathbf{AbGrp} .

By [3, Theorem 3.2]

$$\mathbf{Flow}_{\mathbf{AbGrp}} \cong \mathbf{Mod}_{\mathbb{Z}[t]}. \quad (6.2)$$

This isomorphism of categories is given by the functor $F : \mathbf{Flow}_{\mathbf{AbGrp}} \rightarrow \mathbf{Mod}_{\mathbb{Z}[t]}$, associating to an algebraic flow (G, ϕ) the $\mathbb{Z}[t]$ -module G_ϕ , where G_ϕ is G with the structure of $\mathbb{Z}[t]$ -module given by the multiplication $tx = \phi(x)$ for every $x \in G$. Moreover, for a morphism $u : (G, \phi) \rightarrow (H, \psi)$ in $\mathbf{Flow}_{\mathbf{AbGrp}}$, $F(u) = u : G_\phi \rightarrow H_\psi$ is an homomorphism of $\mathbb{Z}[t]$ -modules. In the opposite direction consider the functor $F' : \mathbf{Mod}_{\mathbb{Z}[t]} \rightarrow \mathbf{Flow}_{\mathbf{AbGrp}}$, which associates to $M \in \mathbf{Mod}_{\mathbb{Z}[t]}$ the algebraic flow (M, μ_t) , where $\mu_t(x) = tx$ for every $x \in M$. If $u : M \rightarrow N$ is an homomorphism in $\mathbf{Mod}_{\mathbb{Z}[t]}$, then $F'(u) = u : (M, \mu_t) \rightarrow (N, \mu_t)$ is a morphism in $\mathbf{Flow}_{\mathbf{AbGrp}}$.

Remark 6.2. By the isomorphism (6.2), every function f defined on endomorphisms of abelian groups can be viewed as a function $f : \mathbf{Flow}_{\mathbf{AbGrp}} \rightarrow \mathbb{R}_+ \cup \{\infty\}$ or equivalently as $f : \mathbf{Mod}_{\mathbb{Z}[t]} \rightarrow \mathbb{R}_+ \cup \{\infty\}$ by letting $f(M) = f(M, \mu_t) = f(\mu_t)$. In particular, this holds for the algebraic entropy $h : \mathbf{Mod}_{\mathbb{Z}[t]} \rightarrow \mathbb{R}_+ \cup \{\infty\}$.

So we have the following

Proposition 6.3. *The algebraic entropy h is a length function of $\mathbf{Mod}_{\mathbb{Z}[t]}$.*

Proof. With respect to Definition 6.1, h satisfies (a) by Example 2.4, (b) by the Addition Theorem 1.3 and (c) by Proposition 2.8(c). \square

It is proved in [27] that the values of a length function L of \mathbf{Mod}_R are determined by its values on the finitely generated R -modules. In case R is a Noetherian commutative ring, L is determined by its values $L(R/\mathfrak{p})$ for prime ideals \mathfrak{p} of R [27, Corollary of Lemma 2].

Second proof of Theorem 1.5. Let $h^* = \{h_G^* : G \text{ abelian group}\}$ be a collection of functions $h_G^* : \text{End}(G) \rightarrow \mathbb{R}_+$ satisfying (a) – (e) from Theorem 1.5. Let $R = \mathbb{Z}[t]$. By Remark 6.2 h^* can be viewed as a function $h^* : \mathbf{Mod}_R \rightarrow \mathbb{R}_+ \cup \{\infty\}$. Then h^* is a length function of \mathbf{Mod}_R by (b) and (c). So in view of Proposition 6.3 and of the above mentioned results from [27] it suffices only to check that

$$h^*(R/\mathfrak{p}) = h(R/\mathfrak{p}) \text{ for all prime ideals } \mathfrak{p} \text{ of } R.$$

Let us recall that R has Krull dimension 2. More precisely, the non-zero prime ideals \mathfrak{p} of R are either minimal or maximal. In particular, if \mathfrak{p} is a minimal prime ideal of R , then $\mathfrak{p} = (f(t))$, where $f(t) \in R$ is irreducible (either $f(t) = p$ is a prime in \mathbb{Z} , or $f(t)$ is irreducible with $\deg f(t) > 0$). On the other hand, a maximal ideal \mathfrak{m} of R is of the form $\mathfrak{m} = (p, f(t))$, where p is a prime in \mathbb{Z} and $f(t) \in R$ has $\deg f(t) > 0$ and is irreducible modulo p .

(i) For $\mathfrak{p} = 0$, we prove that $h^*(R) = \infty = h(R)$.

To this end we have to show that $\mu_t : R \rightarrow R$ has $h^*(\mu_t) = \infty = h(\mu_t)$. Indeed, R is isomorphic to $\mathbb{Z}^{(\mathbb{N})}$ and μ_t is conjugated to $\beta_{\mathbb{Z}}$ through this isomorphism. By Example 2.10 $h(\beta_{\mathbb{Z}}) = \infty$ and so $h(\mu_t) = \infty$ by Proposition 2.8(a). As shown in (iv) of the first proof of Theorem 1.5, also $h^*(\beta_{\mathbb{Z}}) = \infty$ and so $h^*(\mu_t) = \infty$ by (a). In particular, $h^*(\mu_t) = \infty = h(\mu_t)$.

(ii) For $\mathfrak{p} = \mathfrak{m}$ a maximal ideal of R , we see now that $h^*(R/\mathfrak{m}) = 0 = h(R/\mathfrak{m})$.

Indeed, $\mathfrak{m} = (p, f(t))$, where $p \in \mathbb{Z}$ is a prime and $f(t) \in R$ is irreducible modulo p . Moreover, $R/\mathfrak{m} \cong \mathbb{Z}(p)[t]/(f_p(t))$, where $f_p(t)$ is the reduction of $f(t)$ modulo p . This shows that R/\mathfrak{m} is finite. Hence $h^*(R/\mathfrak{m}) = h(R/\mathfrak{m})$ by the Uniqueness Theorem for the algebraic entropy of endomorphisms of torsion abelian groups proved in [5, Theorem 6.1], and $h(R/\mathfrak{m}) = 0$ by Lemma 2.5.

(iii) So it remains to see that $h^*(R/\mathfrak{p}) = h(R/\mathfrak{p})$ when \mathfrak{p} is a minimal prime ideal of R .

Assume first that $\mathfrak{p} = (p)$ for some prime $p \in \mathbb{Z}$. We show that $h^*(R/\mathfrak{p}) = \log p = h(R/\mathfrak{p})$.

Indeed, $R/(p) \cong \mathbb{Z}(p)[t]$, $\mathbb{Z}(p)[t] \cong \mathbb{Z}(p)^{(\mathbb{N})}$ and $\mu_t : \mathbb{Z}(p)[t] \rightarrow \mathbb{Z}(p)[t]$ is conjugated to $\beta_{\mathbb{Z}(p)}$ through this isomorphism. By (a) and (d) $h^*(\mu_t) = h^*(\beta_{\mathbb{Z}(p)}) = \log p$, while Proposition 2.8(a) and Example 2.10 yield $h(\mu_t) = h(\beta_{\mathbb{Z}(p)}) = \log p$.

Suppose now that $\mathfrak{p} = (f(t))$, where $f(t) = a_0 + a_1 t + \dots + a_{n-1} t^{n-1} + a_n t^n \in \mathbb{Z}[t]$ is irreducible with $\deg f(t) = n > 0$. We verify that $h^*(R/\mathfrak{p}) = h(R/\mathfrak{p}) = \sum_{|\alpha_i| > 1} \log |\alpha_i|$, where α_i are the roots of $f(t)$.

Let $M = R/(f(t))$. Moreover let $J = \mathbb{Q}[t]f(t)$ be the principal ideal generated by $f(t)$ in $\mathbb{Q}[t]$ and $D = \mathbb{Q}[t]/J$. Let $\pi : \mathbb{Q}[t] \rightarrow D$ be the canonical projection. Since $J \cap R = (f(t))$, π induces an injective homomorphism $M \rightarrow D$ and we can think without loss of generality that M is a subgroup of D (identifying M with $\pi(R)$). Now $D \cong \mathbb{Q}^n$ as abelian groups. Since $r(M) \geq n$, M is essential in D . Consider $\mu_t : D \rightarrow D$ and $\mu_t \upharpoonright_M = \mu_t : M \rightarrow M$. By Corollary 4.8 $h(\mu_t) < \infty$ and so Corollary 2.14(b) implies $h(\bar{\mu}_t) = 0$, where $\bar{\mu}_t = \mu_t : D/M \rightarrow D/M$ is induced by μ_t . Since D/M is torsion, $h^*(\bar{\mu}_t) = h(\bar{\mu}_t) = 0$ by the Uniqueness Theorem for the algebraic entropy of endomorphisms of torsion abelian groups proved in [5, Theorem 6.1]. By (c), $h^*(\bar{\mu}_t) = 0$ implies $h^*(\mu_t \upharpoonright_M) = h^*(\mu_t)$. Moreover, $h(\mu_t \upharpoonright_M) = h(\mu_t)$ by Corollary 2.14(a). So we have to check that $h^*(\mu_t) = h(\mu_t)$, that is, $h^*(D) = h(D)$. We have $D \cong \mathbb{Q}^n$ and through this isomorphism $\mu_t : D \rightarrow D$ is conjugated to the automorphism ϕ of \mathbb{Q}^n given by the companion matrix

$$C(f) = \begin{pmatrix} 0 & 0 & \dots & 0 & -\frac{a_0}{a_n} \\ 1 & 0 & \dots & 0 & -\frac{a_1}{a_n} \\ 0 & 1 & \ddots & 0 & -\frac{a_2}{a_n} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -\frac{a_{n-1}}{a_n} \end{pmatrix}.$$

In particular, $g(t) = \frac{a_0}{a_n} + \frac{a_1}{a_n} t + \dots + \frac{a_{n-1}}{a_n} t^{n-1} + t^n \in \mathbb{Q}[t]$ is the characteristic polynomial of $C(f)$. Hence (a) and Proposition 2.8(a) give respectively $h^*(\mu_t) = h^*(\phi)$ and $h(\mu_t) = h(\phi)$, while (e) and the Algebraic Yuzvinski Formula (1.4) yield $h^*(\phi) = h(\phi) = \sum_{|\alpha_i| > 1} \log |\alpha_i|$, where α_i are the roots of $g(t)$, which are the same roots of $f(t)$. \square

7 The Bridge Theorem

For an abelian group G the Pontryagin dual \widehat{G} is $\text{Hom}(G, \mathbb{T})$ endowed with the compact-open topology [20]. The Pontryagin dual of an abelian group is compact. Moreover, for an endomorphism $\phi : G \rightarrow G$, its adjoint endomorphism $\widehat{\phi} : \widehat{G} \rightarrow \widehat{G}$ is continuous. For basic properties concerning the Pontryagin duality see [6] and [11]. For a subset A of G , the annihilator of A in \widehat{G} is $A^\perp = \{\chi \in \widehat{G} : \chi(A) = 0\}$, while for a subset B of \widehat{G} , the annihilator of B in G is $B^\perp = \{x \in G : \chi(x) = 0 \text{ for every } \chi \in B\}$.

We recall the definition of the topological entropy following [1]. For a compact topological space X and for an open cover \mathcal{U} of X , let $N(\mathcal{U})$ be the minimal cardinality of a subcover of \mathcal{U} . Since X is compact, $N(\mathcal{U})$ is always finite. Let $H(\mathcal{U}) = \log N(\mathcal{U})$ be the *entropy* of \mathcal{U} . For any two open covers \mathcal{U} and \mathcal{V} of X , let $\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$. Define analogously $\mathcal{U}_1 \vee \dots \vee \mathcal{U}_n$, for open covers $\mathcal{U}_1, \dots, \mathcal{U}_n$ of X . Let $\psi : X \rightarrow X$ be a continuous map and \mathcal{U} an open cover of X . Then $\psi^{-1}(\mathcal{U}) = \{\psi^{-1}(U) : U \in \mathcal{U}\}$. The *topological entropy of ψ with respect to \mathcal{U}* is $H_{\text{top}}(\psi, \mathcal{U}) = \lim_{n \rightarrow \infty} \frac{H(\mathcal{U} \vee \psi^{-1}(\mathcal{U}) \vee \dots \vee \psi^{-n+1}(\mathcal{U}))}{n}$, and the *topological entropy of ψ* is $h_{\text{top}}(\psi) = \sup\{H_{\text{top}}(\psi, \mathcal{U}) : \mathcal{U} \text{ open cover of } X\}$.

In the following fact we collect the basic properties of the topological entropy.

Fact 7.1. *Let K be a compact abelian group and $\psi : K \rightarrow K$ a continuous endomorphism.*

- (a) *If H is another compact group, $\eta : H \rightarrow H$ a continuous endomorphism and ψ and η are conjugated (i.e., there exists a topological isomorphism $\xi : K \rightarrow H$ such that $\eta = \xi \psi \xi^{-1}$), then $h_{\text{top}}(\psi) = h_{\text{top}}(\eta)$.*
- (b) *For every $k \in \mathbb{N}_+$, $h_{\text{top}}(\psi^k) = k \cdot h_{\text{top}}(\psi)$. If ψ is an automorphism, then $h_{\text{top}}(\psi^k) = |k| h_{\text{top}}(\psi)$ for every $k \in \mathbb{Z}$.*

- (c) If K is an inverse limit of closed ψ -invariant subgroups $\{K_i : i \in I\}$, then $h_{\text{top}}(\phi) = \sup_{i \in I} h_{\text{top}}(\phi \upharpoonright_{K_i})$.
- (d) If $K = K_1 \times K_2$ and $\psi = \psi_1 \times \psi_2$ with $\psi_i : K_i \rightarrow K_i$ continuous, $i = 1, 2$, then $h_{\text{top}}(\psi_1 \times \psi_2) = h(\psi_1) + h(\psi_2)$.

As the right Bernoulli shift is a fundamental example for the algebraic entropy, the left Bernoulli shift plays the same role for the topological entropy:

Example 7.2. For any compact abelian group K the (left) Bernoulli shift is ${}_K\beta : K^{\mathbb{N}} \rightarrow K^{\mathbb{N}}$ defined by

$$(x_0, x_1, x_2, \dots) \mapsto (x_1, x_2, x_3, \dots).$$

- (a) It is a well-known fact (see [26]) that $h_{\text{top}}({}_K\beta) = \log |K|$, with the usual convention that $\log |K| = \infty$, if $|K|$ is infinite. In particular, $h_{\text{top}}({}_{\mathbb{Z}(p)}\beta) = \log p$, for every prime p .
- (b) Moreover, ${}_K\beta = \widehat{\beta_{\widehat{K}}}$ (see [4, Proposition 6.1]).

It was proved by Bowen and Peters that an Addition Theorem holds also for the topological entropy of continuous endomorphisms of compact groups:

Theorem 7.3 (Addition Theorem). *Let K be a compact abelian group, $\psi : K \rightarrow K$ a continuous endomorphism, N a closed ψ -invariant subgroup of K and $\overline{\psi} : K/N \rightarrow K/N$ the endomorphism induced by ψ . Then $h_{\text{top}}(\psi) = h_{\text{top}}(\psi \upharpoonright_N) + h_{\text{top}}(\overline{\psi})$.*

The following is the Yuzvinski Formula for the topological entropy.

Theorem 7.4 (Yuzvinski Formula). [31] *For $n \in \mathbb{N}_+$ an automorphism ψ of $\widehat{\mathbb{Q}}^n$ is described by a matrix $A \in GL_n(\mathbb{Q})$. Then*

$$h_{\text{top}}(\psi) = \log s + \sum_{|\alpha_i| > 1} \log |\alpha_i|, \quad (7.1)$$

where α_i are the eigenvalues of A and s is the least common multiple of the denominators of the coefficients of the (monic) characteristic polynomial of A .

Remark 7.5. Let G be an abelian group and $\phi \in \text{End}(G)$. Let $K = \widehat{G}$ and $\psi = \widehat{\phi}$. Let also H be a ϕ -invariant subgroup of G . By the Pontryagin duality, $N = H^\perp$ is a closed ψ -invariant subgroup of K , and $N^\perp = H$. Moreover, we have the following commutative diagrams:

$$\begin{array}{ccccc} H & \hookrightarrow & G & \twoheadrightarrow & G/H \\ \phi \upharpoonright_H \downarrow & & \downarrow \phi & & \downarrow \overline{\phi} \\ H & \hookrightarrow & G & \twoheadrightarrow & G/H \end{array} \quad \begin{array}{ccccc} K/N & \longleftarrow & K & \longleftarrow & N \\ \overline{\psi} \uparrow & & \uparrow \psi & & \uparrow \psi \upharpoonright_N \\ K/N & \longleftarrow & K & \longleftarrow & N \end{array}$$

The second diagram is obtained by the first one applying the Pontryagin duality functor. In particular, $\widehat{K/N} \cong H$ and $\widehat{N} \cong G/H$. Moreover, $\widehat{\psi}$ is conjugated to $\phi \upharpoonright_H$ and $\widehat{\psi \upharpoonright_N}$ is conjugated to $\overline{\phi}$. By Proposition 2.8(a),

- (i) $h(\phi \upharpoonright_H) = h(\widehat{\psi})$,
- (ii) $h(\overline{\phi}) = h(\widehat{\psi \upharpoonright_N})$.

The role of the hyperkernel in the case of endomorphisms of abelian groups, for continuous endomorphisms ψ of compact groups K , is played by the *hyperimage* of ψ defined by $\text{Im}_\infty \psi = \bigcap_{n \in \mathbb{N}} \psi^n(K)$, which is a closed ψ -invariant subgroup of K .

The next result shows that, as far as the computation of the value of the topological entropy of continuous endomorphisms of compact groups is concerned, one can restrict it to surjective endomorphisms.

Lemma 7.6. *Let K be a compact group and $\psi : K \rightarrow K$ a continuous endomorphism. Then $\psi \upharpoonright_{\text{Im}_\infty \psi}$ is surjective and $\text{Im}_\infty \psi$ is the largest closed ψ -invariant subgroup of K with this property. Moreover, $h(\psi) = h(\psi \upharpoonright_{\text{Im}_\infty \psi})$.*

Proof. The inclusion $\psi(\text{Im}_\infty \psi) \subseteq \text{Im}_\infty \psi$ is obvious. If $x \in \text{Im}_\infty \psi$, then for every $n \in \mathbb{N}$ there exists $x_n \in K$ such that $x = \psi^{n+1}(x_n)$. Let $y_n = \psi^n(x_n)$. Then

$$y_n \in \psi^n(K) \text{ and } x = \psi(y_n) \text{ for every } n \in \mathbb{N}. \quad (7.2)$$

For $n \in \mathbb{N}$, let now $Y_n = \{y_n, y_{n+1}, \dots\} \subseteq \psi^n(K)$. Since $\psi^n(K)$ is compact, so closed, $\overline{Y_n} \subseteq \psi^n(K)$ as well. By the compactness of K we have $\bigcap_{n \in \mathbb{N}} \overline{Y_n} \neq \emptyset$. Let $y \in \bigcap_{n \in \mathbb{N}} \overline{Y_n} \subseteq \text{Im}_\infty \psi$. By (7.2) $x = \psi(y)$. Hence $x \in \psi(\text{Im}_\infty \psi)$. This proves that $\psi(\text{Im}_\infty \psi) \supseteq \text{Im}_\infty \psi$, and so $\psi(\text{Im}_\infty \psi) = \text{Im}_\infty \psi$. Equivalently, $\psi \upharpoonright_{\text{Im}_\infty \psi}$ is surjective.

Assume that N is a closed ψ -invariant subgroup of K such that $\psi(N) = N$. Hence obviously $N \subseteq \text{Im}_\infty \psi$.

To prove the equality $h(\psi) = h(\psi \upharpoonright_{\text{Im}_\infty \psi})$, by the Addition Theorem 7.3 it suffices to show that $h(\overline{\psi}) = 0$, where $\overline{\psi} : K/\text{Im}_\infty \psi \rightarrow K/\text{Im}_\infty \psi$ is the endomorphism induced by ψ . This follows from the fact that $\bigcap_{n \in \mathbb{N}} \overline{\psi}^n(K/\text{Im}_\infty \psi) = 0$, due to the definition $\text{Im}_\infty \psi = \bigcap_{n \in \mathbb{N}} \psi^n(K)$. \square

We can now proof the Bridge Theorem 1.6.

Proof of Theorem 1.6. Let $K = \widehat{G}$ and $\psi = \widehat{\phi}$.

(i) It is possible to assume that G is torsion-free (i.e., K is connected). Indeed, consider $t(G)$ and the endomorphism $\overline{\phi} : G/t(G) \rightarrow G/t(G)$ induced by ϕ . Then $c(K) = t(G)^\perp$ and so Remark 7.5 gives

$$\begin{array}{ccccc} t(G) \hookrightarrow G & \twoheadrightarrow & G/t(G) & & K/c(K) \longleftarrow K \longleftarrow c(K) \\ \phi \upharpoonright_{t(G)} \downarrow & & \downarrow \phi & & \downarrow \psi \\ t(G) \hookrightarrow G & \twoheadrightarrow & G/t(G) & & K/c(K) \longleftarrow K \longleftarrow c(K) \end{array}$$

where $\overline{\psi} : K/c(K) \rightarrow K/c(K)$ is the induced endomorphism. By the Addition Theorem 1.3 for the algebraic entropy $h(\phi) = h(\phi \upharpoonright_{t(G)}) + h(\overline{\phi})$. By the Addition Theorem 7.3 for the topological entropy $h(\psi) = h(\psi \upharpoonright_{c(K)}) + h(\overline{\psi})$. By Remark 7.5(i) and by Theorem 1.1, we have $h(\phi \upharpoonright_{t(G)}) = h_{\text{top}}(\overline{\psi})$. By Remark 7.5(ii) $h(\overline{\phi}) = h(\psi \upharpoonright_{c(K)})$. So if we show that $h(\psi \upharpoonright_{c(K)}) = h_{\text{top}}(\psi \upharpoonright_{c(K)})$, this will imply $h(\phi) = h_{\text{top}}(\psi)$.

(ii) We can assume that G is torsion-free of finite rank. Indeed, by (i) we can suppose that G is torsion-free. If there exists $g \in G$ such that $r(V(\phi, g))$ is infinite (in particular, $V(\phi, g) \cong \mathbb{Z}^{(\mathbb{N})}$ and $\phi \upharpoonright_{V(\phi, g)}$ is conjugated to $\beta_{\mathbb{Z}}$), then $h(\phi) \geq h(\phi \upharpoonright_{V(\phi, g)}) = \infty$ by Lemma 2.7 and Corollary 4.8(b). Let $N = V(\phi, g)^\perp$. Then N is a closed ψ -invariant subgroup of K such that $K/N \cong \widehat{V(\phi, g)} \cong \mathbb{T}^{\mathbb{N}}$. Moreover, the induced endomorphism $\overline{\psi} : K/N \rightarrow K/N$ is the adjoint of $\phi \upharpoonright_{V(\phi, g)}$. So, according to Example 7.2(b), $\overline{\psi} : K/N \rightarrow K/N$ is conjugated to $\mathbb{T}\beta$, which has $h_{\text{top}}(\mathbb{T}\beta) = \infty$ by Example 7.2(a). Hence $h_{\text{top}}(\psi) = \infty = h(\phi)$.

Assume now that $r(V(\phi, g))$ is finite for every $g \in G$. Then $r(V(\phi, F))$ is finite for every $F \in [G]^{<\omega}$. By Lemma 2.18 $h(\phi) = \sup_{F \in [G]^{<\omega}} h(\phi \upharpoonright_{V(\phi, F)})$. For every $F \in [G]^{<\omega}$ let $N_F = V(\phi, F)^\perp$. As noted in Remark 7.5, for every $F \in [G]^{<\omega}$, $V(\phi, F) \cong \widehat{K/N_F}$ and $h(\phi \upharpoonright_{V(\phi, F)}) = h(\widehat{\overline{\psi}_F})$, where $\overline{\psi}_F : K/N_F \rightarrow K/N_F$ is the endomorphism induced by ψ . So $K = \varprojlim \{K/N_F : F \in [G]^{<\omega}\}$. If we verify that $h(\widehat{\overline{\psi}_F}) = h_{\text{top}}(\overline{\psi}_F)$ for every $F \in [G]^{<\omega}$, then Proposition 2.8(c) and Fact 7.1(c) give

$$h(\phi) = \sup_{F \in [G]^{<\omega}} h(\phi \upharpoonright_{V(\phi, F)}) = \sup_{F \in [G]^{<\omega}} h(\widehat{\overline{\psi}_F}) = \sup_{F \in [G]^{<\omega}} h_{\text{top}}(\overline{\psi}_F) = h_{\text{top}}(\psi).$$

This shows that we can consider only torsion-free abelian groups of finite rank.

(iii) It suffices to prove the thesis for G a divisible torsion-free abelian group of finite rank. Indeed, by (ii) we can assume that G is a torsion-free abelian group of finite rank $n \in \mathbb{N}_+$, i.e., K is a connected compact abelian group of dimension n . Assume without loss of generality (by Proposition 2.8(a) and Fact 7.1(a)) that $D(G) = \mathbb{Q}^n$, and so that $\widehat{D(G)} = \widehat{\mathbb{Q}}^n$. Let $\varphi = \widetilde{\phi} : \mathbb{Q}^n/G \rightarrow \mathbb{Q}^n/G$ be the induced endomorphism, and $\eta = \widetilde{\phi} : \widehat{\mathbb{Q}}^n \rightarrow \widehat{\mathbb{Q}}^n$, $N = G^\perp$. So Remark 7.5 gives the following corresponding diagrams:

$$\begin{array}{ccccc} G \hookrightarrow \mathbb{Q}^n & \twoheadrightarrow & \mathbb{Q}^n/G & & K \longleftarrow \widehat{\mathbb{Q}}^n \longleftarrow N \\ \phi \downarrow & & \downarrow \widetilde{\phi} & & \downarrow \eta \\ G \hookrightarrow \mathbb{Q}^n & \twoheadrightarrow & \mathbb{Q}^n/G & & K \longleftarrow \widehat{\mathbb{Q}}^n \longleftarrow N \end{array}$$

Then $h(\phi) = h(\widetilde{\phi})$ by Proposition 2.12. The next step is to show that $h_{\text{top}}(\psi) = h_{\text{top}}(\eta)$. To this end, since $h_{\text{top}}(\eta) = h_{\text{top}}(\eta \upharpoonright_N) + h_{\text{top}}(\psi)$ by the Addition Theorem 7.3, it suffices to prove that $h_{\text{top}}(\eta \upharpoonright_N) = 0$. So \mathbb{Q}^n/G is torsion, since G is essential in \mathbb{Q}^n . Therefore, $h(\eta \upharpoonright_N) = h_{\text{top}}(\eta \upharpoonright_N)$ by Theorem 1.1. By Remark

7.5 $h(\varphi) = h(\widehat{\eta \upharpoonright_N})$. Then it remains to verify that $h(\varphi) = 0$. Let $H = \mathbb{Q}^n/G \cong \bigoplus_p \mathbb{Z}(p^\infty)^{k_p}$ with each $k_p \in \mathbb{N}$, $k_p \leq n$. For every $m \in \mathbb{N}_+$, the fully invariant subgroup $H[m] = \{x \in H : mx = 0\}$ of H is finite, so $h(\varphi \upharpoonright_{H[m]}) = 0$. Since $H = \varinjlim H[m]$, Proposition 2.8(c) yields $h(\varphi) = \sup_{m \in \mathbb{N}_+} h(\varphi \upharpoonright_{H[m]}) = 0$.

We have seen that $h(\phi) = h(\tilde{\phi})$ and that $h_{top}(\psi) = h_{top}(\eta)$. Then $h(\tilde{\phi}) = h_{top}(\eta)$ would imply $h(\phi) = h_{top}(\psi)$. In other words, it suffices to prove the thesis for $\phi \in \text{End}(\mathbb{Q}^n)$.

(iv) We can suppose that ϕ is injective (i.e., ψ surjective). Indeed, consider the corresponding diagrams given by Remark 7.5:

$$\begin{array}{ccccc} \ker_\infty \phi & \xrightarrow{\quad} & G & \twoheadrightarrow & G/\ker_\infty \phi \\ \phi \upharpoonright_{\ker_\infty \phi} \downarrow & & \downarrow \phi & & \downarrow \bar{\phi} \\ \ker_\infty \phi & \xrightarrow{\quad} & G & \twoheadrightarrow & G/\ker_\infty \phi \end{array} \qquad \begin{array}{ccccc} K/\text{Im}_\infty \psi & \xleftarrow{\quad} & K & \xleftarrow{\quad} & \text{Im}_\infty \psi \\ \bar{\psi} \uparrow & & \uparrow \psi & & \uparrow \psi \upharpoonright_{\text{Im}_\infty \psi} \\ K/\text{Im}_\infty \psi & \xleftarrow{\quad} & K & \xleftarrow{\quad} & \text{Im}_\infty \psi \end{array}$$

Indeed, $\text{Im}_\infty \psi = (\ker_\infty \phi)^\perp$. By Corollary 2.6 and the Addition Theorem 1.3 $h(\phi) = h(\bar{\phi})$, where the induced endomorphism $\bar{\phi} : G/\ker_\infty \phi \rightarrow G/\ker_\infty \phi$ is injective. By Lemma 7.6 $h_{top}(\psi) = h_{top}(\psi \upharpoonright_{\text{Im}_\infty \psi})$, where $\psi \upharpoonright_{\text{Im}_\infty \psi}$ is surjective. Remark 7.5 yields $h(\bar{\phi}) = h(\psi \upharpoonright_{\text{Im}_\infty \psi})$. So if we prove that $h(\psi \upharpoonright_{\text{Im}_\infty \psi}) = h_{top}(\psi \upharpoonright_{\text{Im}_\infty \psi})$, this will imply $h(\phi) = h_{top}(\psi)$.

(v) By (iii) we can assume that G is a divisible torsion-free abelian group of finite rank, that is, $G = \mathbb{Q}^n$ for $n = r(G)$. By (iv) we can suppose that $\phi \in \text{End}(\mathbb{Q}^n)$ is injective; then ϕ is also surjective and so $\phi \in \text{Aut}(\mathbb{Q}^n)$. Therefore, $h(\phi) = h_{top}(\psi)$ by Theorem 1.2. \square

Note that step (v) of the proof of the Bridge Theorem 1.6 can be proved also applying the Algebraic Yuzvinski Formula (1.4) to ϕ and the Yuzvinski Formula (7.1) to $\tilde{\phi}$. So if one has an independent proof of the Algebraic Yuzvinski Formula (1.4), this would be a proof of the Bridge Theorem 1.6 independent from the particular case proved by Peters.

8 Computation of Peters entropy via Mahler measure

Let $f(t) = a_0 + a_1 t + \dots + a_k t^k \in \mathbb{Z}[t]$ be a primitive polynomial. Let $\{\alpha_i : i = 1, \dots, k\} \subseteq \mathbb{C}$ be the set of all roots of $f(t)$. The *Mahler measure* of $f(t)$ is

$$m(f(t)) = \log |a_k| + \sum_{|\alpha_i| > 1} \log |\alpha_i|.$$

Sometimes the exponential form of Mahler measure $M(f(t)) = \sum_{i=1}^k \max\{1, |\alpha_i|\}$ is also considered [12]; clearly, $m(f(t)) = \log M(f(t))$. The Mahler measure plays an important role in number theory and arithmetic geometry (see [8, Chapter 1]).

If $g(t) \in \mathbb{Q}[t]$ is monic, then there exists a smallest $s \in \mathbb{N}_+$ such that $sg(t) \in \mathbb{Z}[t]$; in particular, $sg(t)$ is primitive. So we can define the Mahler measure of $g(t)$ as $m(g(t)) = m(sg(t))$.

For an algebraic number $\alpha \in \mathbb{C}$, the Mahler measure $m(\alpha)$ of α is the Mahler measure of the minimal polynomial of α .

Lehmer [15], with the aim of generating large primes, associated to any monic polynomial $f(t) \in \mathbb{Z}[t]$ with roots $\alpha_1, \dots, \alpha_k$ the sequence of integers

$$\Delta_n(f(t)) = \prod_{i=1}^k |1 - \alpha_i^n|.$$

The idea comes from Mersenne primes generated by the polynomial $f(t) = t - 2$. Lehmer was using the polynomial $f(t) = t^3 - t - 1$. This is the the non-reciprocal polynomial with the smallest positive Mahler measure [25]. The polynomial

$$g(t) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$$

is the reciprocal polynomial with the smallest known positive Mahler measure, that is, $m(g(t)) = \log \lambda$, where $\lambda = 1.17628\dots$ is the Lehmer number [12]. Still in [12] it is noted that λ is the largest real root of $g(t)$ and it is the only one of its algebraic conjugates outside the unit circle (i.e., λ is a Salem number). If there exists a polynomial $h(t)$ with positive Mahler measure smaller than this, then $\deg h(t) \geq 55$ [17].

Remark 8.1. Let $f(t) \in \mathbb{Z}[t] \setminus \{0\}$ be monic with roots $\alpha_1, \dots, \alpha_k$. Then

$$m(f(t)) = \lim_{n \rightarrow \infty} \frac{\log |\Delta_n(f(t))|}{n}.$$

Indeed,

$$\lim_{n \rightarrow \infty} \frac{\log |\Delta_n(f(t))|}{n} = \lim_{n \rightarrow \infty} \frac{\sum_{i=1}^k \log |1 - \alpha_i^n|}{n} = \sum_{i=1}^k \lim_{n \rightarrow \infty} \frac{\log |1 - \alpha_i^n|}{n} = \sum_{|\alpha_i| > 1} \log |\alpha_i|.$$

Theorem 8.2 (Kronecker Theorem). [14] *Let $f(t) \in \mathbb{Z}[t]$ be a monic polynomial with roots $\alpha_1, \dots, \alpha_k$. If α_1 is not a root of unity, then $|\alpha_i| > 1$ for at least one $i \in \{1, \dots, k\}$.*

Corollary 8.3. *Let $f(t) \in \mathbb{Z}[t] \setminus \{0\}$ be primitive. Then $m(f(t)) = 0$ if and only if $f(t)$ is cyclotomic (i.e., all the roots of $f(t)$ are roots of unity). Consequently, if α is an algebraic integer, then $m(\alpha) = 0$ if and only if α is a root of unity.*

Proof. Let $f(t) = a_0 + a_1 t + \dots + a_k t^k$ and $\deg f(t) = k$. Assume that all roots $\alpha_1, \dots, \alpha_k$ of $f(t)$ are roots of unity. Then there exists $m \in \mathbb{N}_+$ such that every α_i is a root of $t^m - 1$. By Gauss Lemma $f(t)$ divides $t^m - 1$ in $\mathbb{Z}[t]$. Therefore, $a_k = 1$, hence $m(f(t)) = 0$. Suppose now that $m(f(t)) = 0$. Then we can suppose without loss of generality that $f(t)$ is monic. Moreover, all $|\alpha_i| \leq 1$ for every $i \in \{1, \dots, k\}$. By Theorem 8.2 each α_i is a root of unity. \square

This determines completely the case of zero Mahler measure.

Problem 8.4 (Lehmer Problem). [15] *Given any $\delta > 0$, is there an algebraic integer whose Mahler measure is strictly between 0 and δ ?*

This problem is equivalent replacing algebraic integer with algebraic number. Moreover, this is equivalent to ask whether $\inf\{m(f(t)) : f(t) \in \mathbb{Z}[t] \text{ primitive}\} = 0$. It suffices to consider monic polynomials. Indeed, if $\deg(f(t)) = k$ and $f(t) = a_0 + a_1 t + \dots + a_k t^k$ with $|a_k| > 1$, then $m(f(t)) \geq \log |a_k| \geq \log 2 > 0$.

It is known from [5] that the algebraic entropy ent takes values in $\log \mathbb{N}_+ \cup \{\infty\}$. Then

$$\inf\{\text{ent}(\phi) : G \text{ abelian group}, \phi \in \text{End}(G), \text{ent}(\phi) > 0\} = \log 2. \quad (8.1)$$

We consider now the positive values of the algebraic entropy h , that is, the real number

$$\varepsilon = \inf\{h(\phi) : G \text{ abelian group}, \phi \in \text{End}(G), h(\phi) > 0\}.$$

Problem 8.5. *Is $\varepsilon = 0$?*

By the Bridge Theorem 1.6, this problem is equivalent to the major open problem about the infimum of the positive values of the topological entropy (see [28]).

The following is an immediate consequence of the Algebraic Yuzvinski Formula (1.4).

Corollary 8.6. *Let $n \in \mathbb{N}_+$ and $\phi \in \text{Aut}(\mathbb{Q}^n)$. Then $h(\phi) = m(f(t))$, where $f(t) \in \mathbb{Q}[t]$ is the (monic) characteristic polynomial of the matrix associated to ϕ .*

Theorem 8.7. *We have $\varepsilon = \inf\{h(\phi) : n \in \mathbb{N}_+, \phi \in \text{Aut}(\mathbb{Q}^n), h(\phi) > 0\}$.*

Proof. (i) Let G be an abelian group, $\phi \in \text{End}(G)$ and $\bar{\phi} : G/t(G) \rightarrow G/t(G)$ the endomorphism induced by ϕ . Since $h(\phi \upharpoonright_{t(G)}) = \text{ent}(\phi \upharpoonright_{t(G)})$, we can suppose that $h(\phi \upharpoonright_{t(G)}) = 0$, otherwise $h(\phi) \geq h(\phi \upharpoonright_{t(G)}) \geq \log 2$ by Lemma 2.7 and (8.1). By the Addition Theorem 1.3 $h(\phi) = h(\bar{\phi})$. In other words, we can consider only torsion-free abelian groups G .

(ii) By (i) assume that G is a torsion-free abelian group and $\phi \in \text{End}(G)$. If there exists $g \in G$ such that $r(V(\phi, g))$ is infinite, then $h(\phi) \geq h(\phi \upharpoonright_{V(\phi, g)}) = \infty$ by Lemma 2.7 and Corollary 4.8(b). Then we can consider only G such that $r(V(\phi, g))$ is finite for every $g \in G$. Then $r(V(\phi, F))$ is finite for every $F \in [G]^{<\omega}$. By Lemma 2.18 $h(\phi) = \sup_{F \in [G]^{<\omega}} h(\phi \upharpoonright_{V(\phi, F)})$. This shows that we can consider only torsion-free abelian groups of finite rank.

(iii) By (i) assume that G is a torsion-free abelian group and $\phi \in \text{End}(G)$. By Theorem 2.12 $h(\phi) = h(\tilde{\phi})$, and so we can reduce to divisible torsion-free abelian groups.

(iv) By (ii) and (iii) we can consider divisible torsion-free abelian groups G of finite rank, namely, $G \cong \mathbb{Q}^n$ for some $n \in \mathbb{N}_+$. Let $\phi \in \text{End}(\mathbb{Q}^n)$. By Proposition 4.5 $h(\phi) = h(\bar{\phi})$, where the induced endomorphism $\bar{\phi} : \mathbb{Q}^n / \ker_\infty \phi \rightarrow \mathbb{Q}^n / \ker_\infty \phi$ is injective (hence surjective) and $\mathbb{Q}^n / \ker_\infty \phi \cong \mathbb{Q}^m$ for some $m \in \mathbb{N}$, $m \leq n$, as $\ker_\infty \phi$ is pure in \mathbb{Q}^n (so $\mathbb{Q}^n / \ker_\infty \phi$ is divisible and torsion-free by Lemma 4.1(c)). Therefore we can consider automorphisms of \mathbb{Q}^n , and this gives the thesis. \square

Theorem 8.7 and Corollary 8.6 have the following immediate consequence.

Corollary 8.8. *Problem 8.5 is equivalent to Lehmer Problem 8.4.*

By Corollary 8.6 the algebraic entropy $h(\phi)$ of a $\phi \in \mathbb{Q}^n$ is equal to the Mahler measure of a polynomial $f(t) \in \mathbb{Z}[t]$ with non-zero constant term. Now we see the viceversa, that is, the Mahler measure of a polynomial $f(t) \in \mathbb{Z}[t]$ with non-zero constant term is equal to the algebraic entropy of an automorphism ϕ of \mathbb{Q}^n for some $n \in \mathbb{N}_+$. Indeed, let $f(t) = a_0 + a_1 t + \dots + a_k t^k \in \mathbb{Z}[t]$ with $\deg f(t) = k$ and $a_0 \neq 0$. Let

$$C(f) = \begin{pmatrix} 0 & 0 & \dots & 0 & -\frac{a_0}{a_k} \\ 1 & 0 & \dots & 0 & -\frac{a_1}{a_k} \\ 0 & 1 & \ddots & 0 & -\frac{a_2}{a_k} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -\frac{a_{k-1}}{a_k} \end{pmatrix} \quad (8.2)$$

be the companion matrix associated to $f(t)$. The characteristic polynomial of $C(f)$ is $f(t)$. Since $\det C(f) = (-1)^{k+2} \frac{a_0}{a_k} \neq 0$ by hypothesis, $C(f)$ is the matrix associated to an automorphism ϕ of \mathbb{Q}^n . Then $m(f(t)) = h(\phi)$ by Corollary 8.6.

Remark 8.9. Let $\alpha \in \mathbb{C}$ be an algebraic number of degree $n \in \mathbb{N}_+$ over \mathbb{Q} and let $f(t) \in \mathbb{Q}[t]$ be its minimal polynomial, with $\deg f(t) = n$. Let $K = \mathbb{Q}(\alpha)$; then $K \cong \mathbb{Q}^n$ as abelian groups. Following [32], call *algebraic entropy* $h(\alpha)$ of α the algebraic entropy $h(\mu_\alpha)$, where μ_α is the multiplication by α in K . Let a_k be the smallest positive integer such that $a_k f(t) = a_0 + a_1 t + \dots + a_k t^k \in \mathbb{Z}[t]$. With respect to the basis $\{1, \alpha, \dots, \alpha^{k-1}\}$ of K , the matrix associated to μ_α is the companion matrix $C(f)$ in (8.2). The characteristic polynomial of $C(f)$ is $f(t)$. By Corollary 8.6, $h(\mu_\alpha) = m(f(t))$, i.e., $h(\alpha) = m(\alpha)$. This means that the algebraic entropy of an algebraic number is precisely its Mahler measure.

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